$$\min_{x} f(x) \quad \text{along} \quad g(x) = 0$$

$$\min_{x,\lambda} \underbrace{f(x) + \lambda^T g(x)}_{F(x,\lambda)}$$

δx

$\delta \lambda$

$$\delta F = \frac{\partial f}{\partial x}\delta x + \lambda^T \frac{\partial g}{\partial x}\delta x + g(x)\,\delta\lambda = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} + \lambda^T \frac{\partial g}{\partial x} = 0$$

$$\Rightarrow \quad g(x) = 0$$

$$\dot{x} = f(\underset{\uparrow}{x}, \underset{\uparrow}{u}, t)$$

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t)\, dt$$

$$\psi(x(t_f), t_f) = 0$$

$$\tilde{J} = \phi(x(t_f), t_f) + \lambda^T \psi(x(t_f), t_f) + \int_{t_0}^{t} (\underbrace{L(x, u, t) + p^T(f(x, u, t)}_{H \quad \text{Hamiltonian}} - \dot{x})]\, dt$$

t_0 fixed

$\lambda, p, x, u, x(t_f), t_f$ ← ind. variations

$$H(x,u,p,t) = L(x,u,t) + p^T f(x,u,t)$$

$\delta \tilde{J}$? $\frac{\partial H}{\partial x} \triangleq D_1 H$ $\frac{\partial H}{\partial u} \triangleq D_2 H$ ...

$$\delta \tilde{J} = \Big(\overset{1\ \overset{n}{\boxed{\quad\quad}}}{D_1 \phi} + \overset{1\ \overset{n}{\boxed{\quad\quad}}}{D_1 \psi^T \lambda}\Big)\, \overset{\overset{1}{}\ n}{\delta x(t_f)} \leftarrow \delta x(t_f)$$

$$+ \left(D_2^{\square} \phi + D_2^{\square} \psi^T \lambda\right) \delta t_f^{\square} \leftarrow \delta t_f$$

$$+ \overset{1\ \overset{p}{\boxed{\quad\quad}}}{\psi^T}\ \overset{1}{\delta\lambda}\,\big|_p \qquad \delta\lambda$$

$$+ \left(H - p^T \dot{x}\right) \delta t_f$$

$$+ \int_{t_0}^{t_f} \Big(D_1 H\, \delta x + D_3 H\, \delta u - p^T \delta \dot{x} + (D_2 H^T - \dot{x})^T \delta p\Big) dt$$

$$= 0$$

integrate by parts.

$$\delta \tilde{J} = 0 \Rightarrow$$

$$\psi(x(t_f), t_f) = 0 \quad \longleftarrow \delta\lambda$$

state eqn $\quad \dot{x} = \dfrac{\partial H}{\partial p}^T = f(x, u, t) \quad \longleftarrow \delta p$

Hamilton's eqns.

co-state eqn. $\quad \dot{p} = -\dfrac{\partial H}{\partial x}^T \quad \longleftarrow \delta x$

input stationary $\quad \dfrac{\partial H}{\partial u} = 0 \quad \longleftarrow \delta u$

boundary conditions $\quad \begin{cases} D_1 \phi - p^T = -D_1 \psi^T \lambda \big|_{t_f} & \delta x(t_f) \longleftarrow \\ H + D_2 \phi = -D_2 \psi^T \lambda \big|_{t_f} & \delta t_f \end{cases}$

necessary conditions for optimality

Suff. cond for opt:

$$\underbrace{D_2^2 H(x, u, p, t)}_{\text{Hessian}} \text{ pos. definite}$$

p9. lecture notes ..

3. LQR

$$\dot{x} = \overset{\swarrow}{A} x + \overset{\swarrow}{B} u \qquad T = t_f$$

$$J = \frac{1}{2} (x(T)^T \overset{\text{psd}}{Q_T} x(T) + \int_{t_0}^{T} (x(\tau)^T \overset{\text{psd}}{Q} x(\tau) + u(\tau)^T \overset{\text{pd}}{R} u(\tau)) d\tau$$

in general:

$$(x(T) - x_{des})^T Q_T (x(T) - x_{des})) \;\cdots$$

$$H(x, u, p) = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + p^T (Ax + Bu)$$

$$\dot{x} = Ax + Bu$$

$$x(t_0) = x_0$$

$$\dot{p} = -A^T p - Qx \qquad \dot{p} = -\frac{\partial H}{\partial x}^T \quad (\delta x)$$

$$\boxed{u(t) = -R^{-1} B^T p(t)} \Leftarrow 0 = Ru + B^T p \qquad \frac{\partial H}{\partial u} = 0 \quad (\delta u)$$

$$p(T) = Q_T \, x(T)$$

$\uparrow$ nec. cond.

$$R = D_2 H(x, u, p) > 0$$

$\uparrow$ ✓

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^{T} \\ -Q & -A^{T} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \qquad \begin{aligned} x(t_0) &= x_0 \\ p(T) &= Q_T\, x(T) \end{aligned}$$

$\exists$ V pd which solves.

Riccati Eqn

$$-\dot{V} = -A^{T}V + VA - VBR^{-1}B^{T}V + Q$$

$$V(T) = Q_T$$

such that $p(t) = V(t)x(t)$

proof: $p = Vx$ satisfies $\dot{p} = -Qx - A^{T}p.$

$V(T) = Q_T$

Sonil's lecture for EE221A: sampling time

$$x_{t+1} \overset{\Delta}{=} x(t+1) \overset{\Delta}{=} x((t+1)T)$$

$$x_t \qquad x(t)$$

$$x_{t+1} = Ax_t + Bu_t, \quad x_0$$

$$t \in \{0, 1, \dots N\}$$

$$U = (u_0, u_1, \dots u_{N-1})$$

$$J(U, x_0) = x_N^T Q_f x_N + \sum_{\tau=0}^{N-1} (x_\tau^T Q x_\tau + u_\tau^T R u_\tau$$

suppose you start @ t, not 0

cost-to-go: $J(U_t, x_t) = x_N^T Q_f x_n + \sum_{\tau=t}^{N-1}(x_\tau^T Q x_\tau + u_\tau^T R u_\tau)$

$$U_t = (u_t, u_{t+1} \dots u_{N-1})$$

as a function of x_{t+1}

$$J_t^*(x_t) = \min_{u_t} \Big(x_t^\top Q x_t + u_t^\top R u_t + J_{t+1}^*(Ax_t + Bu_t)\Big)$$

Theorem 1

$$J_t^*(z) = z^\top P_t z$$

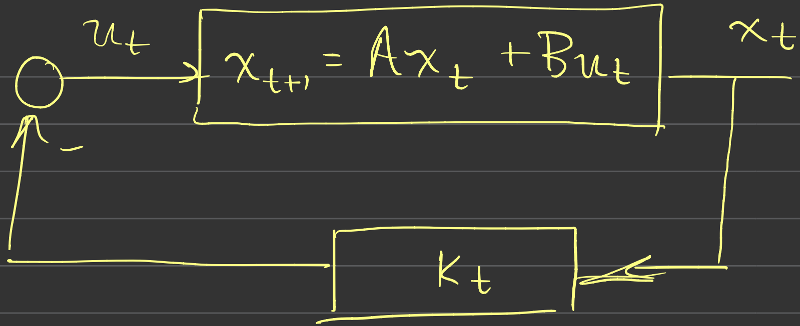
$$u_t^* = -K_t z$$

where

$$P_t = Q + K_t^\top R K_t + (A - BK_t)^\top P_{t+1} (A - BK_t)$$

$$P_N = Q_f.$$

$$K_t = (R + B^\top P_{t+1} B)^{-1} B^\top P_{t+1} A.$$



The best one is linear state feedback

See §3.2 for an alternate approach to DT LQR, by discretizing the CT LQR problem that we solved earlier.

DT LQR computation .... §3.3 of lecture notes.

$$x_{t+1} = A_t x_t + B_t u_t + c_t$$

$$J = \frac{1}{2} \Big(x_N^T Q_T x_N + 2 q_N^T x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + 2 u_k^T S_k x_k + u_k^T R u_k + 2 q_k^T x + 2 r_k^T u_k \Big)$$

$$z = [\, x_0, u_0, \cdots\cdots x_{N-1}, u_{N-1}, x_N \,]^T$$

$$J = \frac{1}{2} z^T H z + h^T z$$

$H, h.$

$Gz + g = 0 \qquad G, g.$

$$\min_{z} \ \frac{1}{2} z^T H z + h^T z \quad \text{st} \quad Gz + g = 0$$