EECS208 Written HW2

Issued: Sep. 22. **Due:** Oct. 3, 11:59 PM via Gradescope

Reading: Chapters 3 and Appendix A.8 - A.9 of *High-Dim Data Analysis with Low-Dim Models*.

Problem 1 (Singular Values of Matrices)

Exercise 3.4 of *High-Dim Data Analysis with Low-Dim Models.*

Problem 2 (Singular Values, Spectral Norm, and Frobenius Norm)

Exercise 3.5 of *High-Dim Data Analysis with Low-Dim Models.*

Problem 3 (Incoherence and Singular Values, Exercise 3.6)

Given a matrix $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n] \in \mathbb{R}^{m \times n}$, whose column vectors have unit length $\|\boldsymbol{a}_i\|_2 = 1, \forall i \in [n]$. Let $\mu(\boldsymbol{A})$ denote the mutual the mutual coherence of $\boldsymbol{A}$ and let $\mathsf{krank}(\boldsymbol{A})$ denote the Kruskal rank of $\boldsymbol{A}$. $\forall k \leq \mathsf{krank}(\boldsymbol{A})$, suppose we pick k column vectors $\boldsymbol{a}_{i_1}, \boldsymbol{a}_{i_2}, \ldots \boldsymbol{a}_{i_k}$ from $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$, and let $\boldsymbol{A}_{\mathsf{I}} = [\boldsymbol{a}_{i_1}, \boldsymbol{a}_{i_2}, \ldots \boldsymbol{a}_{i_k}]$ be the submatrix formed by the selected column vectors. Show that

$$1 - k\mu(\boldsymbol{A}) \leq \sigma_{\min}(\boldsymbol{A}_{\mathsf{I}}^\top \boldsymbol{A}_{\mathsf{I}}) \leq \sigma_{\max}(\boldsymbol{A}_{\mathsf{I}}^\top \boldsymbol{A}_{\mathsf{I}}) \leq 1 + k\mu(\boldsymbol{A}), \tag{0.1}$$

where we use $\sigma_{\min}(\boldsymbol{M}), \sigma_{\max}(\boldsymbol{M})$ to denote the minimum and maximum singular values of matrix $\boldsymbol{M}$, respectively.

Problem 4 (Singular Values of the Inverse)

Exercise 3.16 of *High-Dim Data Analysis with Low-Dim Models.*

Problem 5 (RSC implies the nullspace property Exercise 3.12)

Prove the following lemma (Lemma 3.15):

Lemma 0.1 *Suppose that $\boldsymbol{A}$ satisfies the restricted strong convexity condition of order k with constant $\alpha \geq 1$, for some $\mu > 0$. Then for any $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_o$, with $\|\boldsymbol{x}_o\|_0 \leq k$, $\boldsymbol{x}_o$ is the unique optimal solution to the ℓ_1 problem:*

$$\min \|\boldsymbol{x}\|_1, \quad \text{subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}. \tag{0.2}$$