EECS208 Written HW1

Issued: Sep. 1. **Due:** Sep. 12, 11:59 PM via Gradescope

Reading: Chapters 1, 2, and Appendix A of *High-Dim Data Analysis with Low-Dim Models*.

Problem 1 (ℓ^p-norm)

Given $p \geq 0$, define the function $\|\cdot\|_p : \mathbb{R}^n \mapsto \mathbb{R}$ as

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p},$$

where we slightly abuse the notation by defining $\|x\|_0 = \sum_{i=1}^n |x_i|^0$ and $\|x\|_\infty = \max_{i \in [n]} |x_i|$. Prove that

1. $\forall p \in [0, 1)$, $\|\cdot\|_p$ is *not* a norm of $\mathbb{R}^n$;
2. $\forall p \in \{1, 2, \infty\}$, $\|\cdot\|_p$ is a norm of $\mathbb{R}^n$.

Problem 2 (Rank-Nullity Theorem)

Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, prove the following statements, and suppose bilinear form of the orthogonal complement $\perp$ is defined via Euclidean inner product (Suppose $\mathbb{V} \subseteq \mathbb{R}^n$ is a linear subspace and $\mathbb{V}^\perp$ is the orthogonal complement of $\mathbb{V}$ in $\mathbb{R}^n$, then we have $\langle v, v^\perp \rangle \doteq \sum_{i=1}^n v_i v_i^\perp = 0, \forall v \in \mathbb{V}, v^\perp \in \mathbb{V}^\perp$). Prove that:

1. $\mathsf{null}(\boldsymbol{A})^\perp = \mathsf{range}(\boldsymbol{A}^\top)$
2. $\mathsf{null}(\boldsymbol{A}^\top) = \mathsf{null}(\boldsymbol{A}\boldsymbol{A}^\top)$
3. $\mathsf{dim}(\mathsf{row}(\boldsymbol{A})) + \mathsf{dim}(\mathsf{null}(\boldsymbol{A})) = n$

Problem 3 (Eigenvalues and Eigenvectors)

Exercise 1.6 of *High-Dim Data Analysis with Low-Dim Models.*

Problem 4 (Ridge Regression)

Exercise 1.8 of *High-Dim Data Analysis with Low-Dim Models.*

Problem 5 (Implicit Bias of Gradient Descent)

Exercise 2.10 of *High-Dim Data Analysis with Low-Dim Models.*