Computational Principles for High-dim Data Analysis

(Lecture Fourteen)

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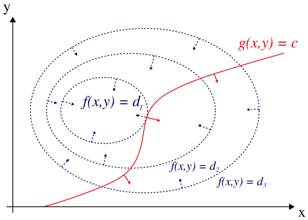
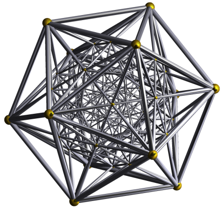




Constrained Convex Optimization
for Structured Data Recovery

"Since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear."

– Leonhard Euler

Optimization Challenges for Structured Data Recovery

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} F(\boldsymbol{x}) \doteq \underbrace{f(\boldsymbol{x})}_{\text{smooth convex}} + \underbrace{g(\boldsymbol{x})}_{\text{nonsmooth convex}}. \tag{1}$$

- **Challenge of Scale**: scale algorithms to when n is very large.

$$\text{Second order methods} \implies \text{First order methods...} \tag{2}$$

- **Nonsmoothness**: first order methods are slow for nonsmooth.

$$O(1/\sqrt{k}) \implies O(1/k) \implies O(1/k^2)... \tag{3}$$

- **Equality Constraints**: augmented Lagrange multiplier (ALM).
- **Separable Structures**: alternating direction of multipliers method (ADMM).

Linear Equality Constrained Optimization

Problem:

$$\min_{\boldsymbol{x}} \; g(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}, \tag{4}$$

where

- $g : \mathbb{R}^n \to \mathbb{R}$ is a (probably nonsmooth) convex function,
- $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{y} \in \mathrm{range}(\boldsymbol{A})$ (so that the problem is feasible).

A Natural Attempt: solve the unconstrained by penalizing the constraint:

$$\hat{\boldsymbol{x}}(\mu) = \arg\min_{\boldsymbol{x}} \; g(\boldsymbol{x}) + \frac{\mu}{2} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2^2 \quad \text{for a large } \mu. \tag{5}$$

- **Pros:** As $\mu \to +\infty$, $\hat{\boldsymbol{x}}(\mu) \to \boldsymbol{x}_\star$ (the "continuation method").
- **Cons:** The rate of convergence depends on $L = \mu\|\boldsymbol{A}\|_2^2$.

Lagrange Multiplier Method

A More Principled Approach:

Definition (The Lagrange Duality)

The *Lagrangian* function of the constrained problem (4):

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) \doteq g(\boldsymbol{x}) + \langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x} - \boldsymbol{y} \rangle, \tag{6}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^m$ is a vector of *Lagrange multipliers*. This gives a *dual function*:

$$d(\boldsymbol{\lambda}) \doteq \inf_{\boldsymbol{x}} \; g(\boldsymbol{x}) + \langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x} - \boldsymbol{y} \rangle. \tag{7}$$

Fact (credited to Lagrange): $\exists \lambda_\star$ such that the optimal solution $(\boldsymbol{x}_\star, \boldsymbol{\lambda}_\star)$ is a saddle point of the Lagrangian:

$$\sup_{\boldsymbol{\lambda}} \inf_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \sup_{\boldsymbol{\lambda}} \inf_{\boldsymbol{x}} \; g(\boldsymbol{x}) + \langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x} - \boldsymbol{y} \rangle = \sup_{\boldsymbol{\lambda}} d(\boldsymbol{\lambda}). \tag{8}$$

Dual Ascent Algorithm for the Lagrangian

Fact: If

$$\boldsymbol{x}'(\boldsymbol{\lambda}) = \arg\min_{\boldsymbol{x}} \; g(\boldsymbol{x}) + \langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x} - \boldsymbol{y} \rangle,$$

then $\boldsymbol{A}\boldsymbol{x}'(\boldsymbol{\lambda}) - \boldsymbol{y}$ is a supergradient $\partial d(\boldsymbol{\lambda})$ of the concave dual $d(\boldsymbol{\lambda})$ at $\boldsymbol{\lambda}$.

(**Why?** Actually this is true for the dual function of general constraints $\boldsymbol{h}(\boldsymbol{x}) = \mathbf{0}$: $d(\boldsymbol{\lambda}) = \min_{\boldsymbol{x}} g(\boldsymbol{x}) + \boldsymbol{\lambda}^T \boldsymbol{h}(\boldsymbol{x})$.)

A Natural Attempt to find the saddle point $(\boldsymbol{x}_\star, \boldsymbol{\lambda}_\star)$ is via *dual ascent*:

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \arg\min_{\boldsymbol{x}} \; \mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}_k), && (9) \\ \boldsymbol{\lambda}_{k+1} &= \boldsymbol{\lambda}_k + t_{k+1}(\boldsymbol{A}\boldsymbol{x}_{k+1} - \boldsymbol{y}). && (10) \end{aligned}$$

- For certain problem classes, this converges to the optimal $(\boldsymbol{x}_\star, \boldsymbol{\lambda}_\star)$.
- However, unfortunately it fails for problems in our settings.

An Example of Failure

Consider the basis pursuit problem:

$$\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}. \tag{11}$$

We can show that

$$\inf_{\boldsymbol{x}} \; \|\boldsymbol{x}\|_1 + \langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x} - \boldsymbol{y} \rangle \; = \begin{cases} -\infty & \|\boldsymbol{A}^*\boldsymbol{\lambda}\|_\infty > 1, \\ -\langle \boldsymbol{\lambda}, \boldsymbol{y} \rangle & \|\boldsymbol{A}^*\boldsymbol{\lambda}\|_\infty \leq 1. \end{cases} \tag{12}$$

Whenever the dual ascent step (10) happens to produce a $\boldsymbol{\lambda}$ such that $\|\boldsymbol{A}^*\boldsymbol{\lambda}\|_\infty > 1$, the algorithm will break down.

The reason is $g(x)$ is not "strongly" convex enough.

Augmented Lagrange Multiplier

One way out: combining (5) and (4), consider the Augmented Lagrangian [Hestenes'69, Powell'69]:

$$\mathcal{L}_\mu(\boldsymbol{x}, \boldsymbol{\lambda}) \doteq g(\boldsymbol{x}) + \langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x} - \boldsymbol{y} \rangle + \frac{\mu}{2} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2^2 . \tag{13}$$

Can be regarded as the Lagrangian for the penalized constrained problem

$$\min_{\boldsymbol{x}} g(\boldsymbol{x}) + \frac{\mu}{2} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2^2 \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}, \tag{14}$$

which has the same optimal solution as the un-penalized problem.

Augmented Lagrange Multiplier

Apply dual ascent to $\mathcal{L}_\mu(\boldsymbol{x}, \boldsymbol{\lambda})$ with a particular step size $t_{k+1} = \mu$,

$$\begin{aligned} \boldsymbol{x}_{k+1} &\in \arg\min_{\boldsymbol{x}} \; \mathcal{L}_\mu(\boldsymbol{x}, \boldsymbol{\lambda}_k), && (15)\\ \boldsymbol{\lambda}_{k+1} &= \boldsymbol{\lambda}_k + \mu\,(\boldsymbol{A}\boldsymbol{x}_{k+1} - \boldsymbol{y}). && (16) \end{aligned}$$

Fact: $\boldsymbol{x}_{k+1}$ always minimizes the unaugmented Lagrangian $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}_{k+1})$ at $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{k+1}$, because:

$$\begin{aligned} \mathbf{0} &\in \partial \mathcal{L}_\mu(\boldsymbol{x}_{k+1}, \boldsymbol{\lambda}_k),\\ &= \partial g(\boldsymbol{x}_{k+1}) + \boldsymbol{A}^*\boldsymbol{\lambda}_k + \mu \boldsymbol{A}^*(\boldsymbol{A}\boldsymbol{x}_{k+1} - \boldsymbol{y}),\\ &= \partial g(\boldsymbol{x}_{k+1}) + \boldsymbol{A}^*\boldsymbol{\lambda}_{k+1},\\ &= \partial \mathcal{L}(\boldsymbol{x}_{k+1}, \boldsymbol{\lambda}_{k+1}). \end{aligned}$$

$\boldsymbol{\lambda}_{k+1}$ is always feasible, no bad behaviors!

Augmented Lagrange Multiplier

Augmented Lagrange Multipler (ALM)

Problem Class: $\min_{\boldsymbol{x}} g(\boldsymbol{x})$ subject to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$.
with $g : \mathbb{R}^n \to \mathbb{R}$ convex and coercive, $\boldsymbol{y} \in \mathrm{range}(\boldsymbol{A})$.

Basic Iteration: set

$$\mathcal{L}_\mu(\boldsymbol{x}, \boldsymbol{\lambda}) = g(\boldsymbol{x}) + \langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x} - \boldsymbol{y} \rangle + \frac{\mu}{2} \left\| \boldsymbol{A}\boldsymbol{x} - \boldsymbol{y} \right\|_2^2 .$$

Repeat:

$$\begin{aligned} \boldsymbol{x}_{k+1} &\in \arg\min_{\boldsymbol{x}} \ \mathcal{L}_\mu(\boldsymbol{x}, \boldsymbol{\lambda}_k), \\ \boldsymbol{\lambda}_{k+1} &= \boldsymbol{\lambda}_k + \mu \left(\boldsymbol{A}\boldsymbol{x}_{k+1} - \boldsymbol{y} \right). \end{aligned}$$

Convergence Guarantee:

$\{\boldsymbol{x}_k\}$ converges to an optimal solution at a rate $O(1/k)$.

ALM for Basis Pursuit

Augmented Lagrange Multipler (ALM) for BP

Problem: $\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1$ subject to $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$,
given $\boldsymbol{y} \in \mathbb{R}^m$ and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
Input: $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\boldsymbol{\lambda}_0 \in \mathbb{R}^m$, and $\beta > 1$.
for $(k = 0, 1, 2, \ldots, K-1)$ **do**
  $\boldsymbol{x}_{k+1} \leftarrow \arg\min \mathcal{L}_{\mu_k}(\boldsymbol{x}, \boldsymbol{\lambda}_k)$ using APG.
  $\boldsymbol{\lambda}_{k+1} \leftarrow \boldsymbol{\lambda}_k + \mu_k(\boldsymbol{A}\boldsymbol{x}_{k+1} - \boldsymbol{y})$.
  $\mu_{k+1} \leftarrow \min\{\beta\mu_k, \mu_{\max}\}$.
end for
Output: $\boldsymbol{x}_\star \leftarrow \boldsymbol{x}_K$.

ALM for Principal Component Pursuit

Augmented Lagrange Multipler (ALM) for PCP

1: **Problem:** $\min_{\boldsymbol{L},\boldsymbol{S}} \|\boldsymbol{L}\|_* + \lambda\|\boldsymbol{S}\|_1$ subject to $\boldsymbol{L} + \boldsymbol{S} = \boldsymbol{Y}$, given $\boldsymbol{Y}$ and $\lambda > 0$.
2: **Input:** $\boldsymbol{L}_0, \boldsymbol{S}_0, \boldsymbol{\Lambda}_0 \in \mathbb{R}^{m\times n}$ and $\beta > 1$.
3: **for** $(k = 0, 1, 2, \ldots, K-1)$ **do**
4: $\quad \{\boldsymbol{L}_{k+1}, \boldsymbol{S}_{k+1}\} \leftarrow \arg\min \mathcal{L}_{\mu_k}(\boldsymbol{L}, \boldsymbol{S}, \boldsymbol{\Lambda}_k)$ using APG.
5: $\quad \boldsymbol{\Lambda}_{k+1} \leftarrow \boldsymbol{\Lambda}_k + \mu_k(\boldsymbol{L}_{k+1} + \boldsymbol{S}_{k+1} - \boldsymbol{Y})$.
6: $\quad \mu_{k+1} \leftarrow \min\{\beta\mu_k, \mu_{\max}\}$.
7: **end for**
8: **Output:** $\boldsymbol{L}_\star \leftarrow \boldsymbol{L}_K, \boldsymbol{S}_\star \leftarrow \boldsymbol{S}_K$.

Optimization with Separable Structures

Example: Principal Component Pursuit

$$\min_{\boldsymbol{L},\boldsymbol{S}} \|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}\|_1 \quad \text{subject to} \quad \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{Y}. \tag{17}$$

A general two-term separable optimization program:

$$\min_{\boldsymbol{x},\boldsymbol{z}} g(\boldsymbol{x}) + h(\boldsymbol{z}) \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} = \boldsymbol{y}, \tag{18}$$

where g and h are convex functions, and $\boldsymbol{y} \in \mathrm{range}([\boldsymbol{A} \mid \boldsymbol{B}])$.

The Lagrangian $\mathcal{L}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\lambda})$ is:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\lambda}) = g(\boldsymbol{x}) + h(\boldsymbol{z}) + \langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{y} \rangle. \tag{19}$$

Optimization with Separable Structures

The augmented Lagrangian $\mathcal{L}_\mu(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\lambda})$ is:

$$\mathcal{L}_\mu(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\lambda}) = g(\boldsymbol{x}) + h(\boldsymbol{z}) + \langle \boldsymbol{\lambda}, \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{y} \rangle + \frac{\mu}{2} \left\| \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{z} - \boldsymbol{y} \right\|_2^2 . \tag{20}$$

The alternating directions method of multipliers (ADMM) conducts a simple, alternating iteration:

$$\begin{aligned} \boldsymbol{z}_{k+1} &\in \arg\min_{\boldsymbol{z}} \mathcal{L}_\mu(\boldsymbol{x}_k, \boldsymbol{z}, \boldsymbol{\lambda}_k), && (21) \\ \boldsymbol{x}_{k+1} &\in \arg\min_{\boldsymbol{x}} \mathcal{L}_\mu(\boldsymbol{x}, \boldsymbol{z}_{k+1}, \boldsymbol{\lambda}_k), && (22) \\ \boldsymbol{\lambda}_{k+1} &= \boldsymbol{\lambda}_k + \mu \left(\boldsymbol{A}\boldsymbol{x}_{k+1} + \boldsymbol{B}\boldsymbol{z}_{k+1} - \boldsymbol{y} \right). && (23) \end{aligned}$$

This is also known as the *Gauss-Seidel iteration*.

ADMM converges at a rate of $O(1/k)$.

ADMM for Principal Component Pursuit

$$\textbf{PCP:} \quad \min_{\boldsymbol{L},\boldsymbol{S}} \|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}\|_1 \quad \text{subject to} \quad \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{Y}. \tag{24}$$

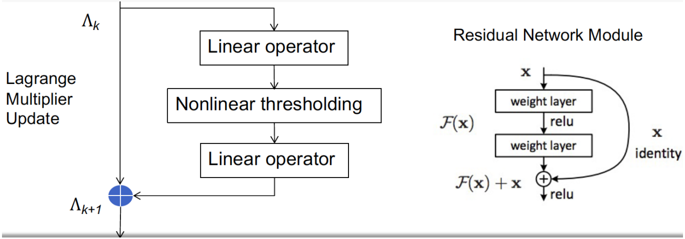
The augmented Lagrangian is

$$\mathcal{L}_\mu(\boldsymbol{L}, \boldsymbol{S}, \boldsymbol{\Lambda}) = \|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}\|_1 + \langle \boldsymbol{\Lambda}, \boldsymbol{L}+\boldsymbol{S}-\boldsymbol{Y} \rangle + \frac{\mu}{2} \|\boldsymbol{L}+\boldsymbol{S}-\boldsymbol{Y}\|_F^2. \tag{25}$$

$$\begin{aligned}
\boldsymbol{L}_{k+1} &= \arg\min_{\boldsymbol{L}} \mathcal{L}_\mu(\boldsymbol{L}, \boldsymbol{S}_k, \boldsymbol{\Lambda}_k) \\
&= \arg\min_{\boldsymbol{L}} \|\boldsymbol{L}\|_* + \frac{\mu}{2} \left\|\boldsymbol{L} + \boldsymbol{S}_k - \boldsymbol{Y} + \mu^{-1}\boldsymbol{\Lambda}_k\right\|_F^2 + \varphi(\boldsymbol{S}_k, \boldsymbol{\Lambda}_k) \\
&= \operatorname{prox}_{\mu^{-1}\|\cdot\|_*}\left[\boldsymbol{Y} - \boldsymbol{S}_k - \mu^{-1}\boldsymbol{\Lambda}_k\right]. \qquad (26) \\
\boldsymbol{S}_{k+1} &= \arg\min_{\boldsymbol{S}} \mathcal{L}_\mu(\boldsymbol{L}_{k+1}, \boldsymbol{S}, \boldsymbol{\Lambda}_k) \\
&= \arg\min_{\boldsymbol{S}} \lambda\|\boldsymbol{S}\|_1 + \frac{\mu}{2} \left\|\boldsymbol{S} + \boldsymbol{L}_{k+1} - \boldsymbol{Y} + \mu^{-1}\boldsymbol{\Lambda}_k\right\|_F^2 + \varphi(\boldsymbol{L}_{k+1}, \boldsymbol{\Lambda}_k) \\
&= \operatorname{prox}_{\lambda\mu^{-1}\|\cdot\|_1}\left[\boldsymbol{Y} - \boldsymbol{L}_{k+1} - \mu^{-1}\boldsymbol{\Lambda}_k\right]. \qquad (27)
\end{aligned}$$

ADMM Algorithm for PCP

1: **Problem:** $\min_{\boldsymbol{L},\boldsymbol{S}} \mathcal{L}_\mu(\boldsymbol{L},\boldsymbol{S},\boldsymbol{\Lambda})$, given $\boldsymbol{Y}$, $\lambda, \mu > 0$.
2: **Input:** $\boldsymbol{L}_0, \boldsymbol{S}_0, \boldsymbol{\Lambda}_0 \in \mathbb{R}^{m\times n}$.
3: **for** $(k = 0, 1, 2, \ldots, K-1)$ **do**
4: $\quad \boldsymbol{L}_{k+1} \leftarrow \operatorname{prox}_{\mu^{-1}\|\cdot\|_*}\left[\boldsymbol{Y} - \boldsymbol{S}_k - \mu^{-1}\boldsymbol{\Lambda}_k\right]$.
5: $\quad \boldsymbol{S}_{k+1} \leftarrow \operatorname{prox}_{\lambda\mu^{-1}\|\cdot\|_1}\left[\boldsymbol{Y} - \boldsymbol{L}_{k+1} - \mu^{-1}\boldsymbol{\Lambda}_k\right]$.
6: $\quad \boldsymbol{\Lambda}_{k+1} \leftarrow \boldsymbol{\Lambda}_k + \mu(\boldsymbol{L}_{k+1} + \boldsymbol{S}_{k+1} - \boldsymbol{Y})$.
7: **end for**
8: **Output:** $\boldsymbol{L}_\star \leftarrow \boldsymbol{L}_K; \boldsymbol{S}_\star \leftarrow \boldsymbol{S}_K$.



Multiple Separable Terms and Consensus Optimization

Machine Learning: Minimizing loss $\sum_i L(\boldsymbol{y}_i, \boldsymbol{x})$ over samples $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_p$. We can partition the data to N batches, each on a machine:

$$\min_{\boldsymbol{x}} \sum_{j=1}^{N} f_j(\boldsymbol{x}) \quad \text{with} \quad f_j(\boldsymbol{x}) = \sum_{i \in \mathsf{I}_j} L(\boldsymbol{y}_i, \boldsymbol{x}). \tag{28}$$

Convert to a consensus problem with separable variables:

$$\min_{\{\boldsymbol{x}_j\}} \sum_{j=1}^{N} f_j(\boldsymbol{x}_j) \quad \text{subject to} \quad \boldsymbol{x}_j = \boldsymbol{z}, \quad j = 1, \ldots, N. \tag{29}$$

Augmented Lagrangian:

$$\mathcal{L}_\mu(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\lambda}) = \sum_{j=1}^{N} f_j(\boldsymbol{x}_j) + \langle \boldsymbol{\lambda}_j, \boldsymbol{x}_j - \boldsymbol{z} \rangle + \frac{\mu}{2} \|\boldsymbol{x}_j - \boldsymbol{z}\|_2^2. \tag{30}$$

Multiple Separable Terms and Consensus Optimization

Apply ADMM to the augmented Lagrangian $\mathcal{L}_\mu(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\lambda})$:

$$\begin{aligned}
\boldsymbol{x}_{j,k+1} &= \arg\min_{\boldsymbol{x}_j} \left\{ f_j(\boldsymbol{x}_j) + \frac{\mu}{2} \left\| \boldsymbol{x}_j - \boldsymbol{z}_k + \frac{1}{\mu} \boldsymbol{\lambda}_{j,k} \right\|_2^2 \right\}, \ \text{(parallel)} && (31) \\
\boldsymbol{z}_{k+1} &= \frac{1}{N} \sum_{j=1}^{N} \left(\boldsymbol{x}_{j,k+1} + \frac{1}{\mu} \boldsymbol{\lambda}_{j,k} \right), \ \text{(aggregate)} && (32) \\
\boldsymbol{\lambda}_{j,k+1} &= \boldsymbol{\lambda}_{j,k} + \mu \left(\boldsymbol{x}_{j,k+1} - \boldsymbol{z}_{k+1} \right). \ \text{(broadcast)} && (33)
\end{aligned}$$

ADMM for ALM is well suited for distributed implementation!

Note: there are many other variants to further improve efficiency and scalability: **accelerated, asynchronous, stochastic**... but convergence guarantee is not a picnic.

Frank-Wolfe Algorithm

Optimizing a smooth, convex function over a *compact* convex set:

$$\min_{\boldsymbol{x}} f(\boldsymbol{x}), \quad \text{subject to} \quad \boldsymbol{x} \in \mathsf{C} \tag{34}$$

which has a finite diameter:

$$\mathsf{diam}(\mathsf{C}) \doteq \max\{\left\|\boldsymbol{x} - \boldsymbol{x}'\right\|_2 \mid \boldsymbol{x}, \boldsymbol{x}' \in \mathsf{C}\}. \tag{35}$$

Two examples:

- Sparse vector recovery:

$$\min_{\boldsymbol{x}} \tfrac{1}{2} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2^2, \quad \text{subject to} \quad \|\boldsymbol{x}\|_1 \leq \tau. \tag{36}$$

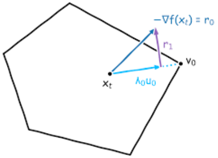
- Low-rank matrix completion:

$$\min_{\boldsymbol{X}} \tfrac{1}{2} \|\mathcal{P}_\Omega[\boldsymbol{X}] - \boldsymbol{Y}\|_F^2, \quad \text{subject to} \quad \|\boldsymbol{X}\|_* \leq \tau. \tag{37}$$

Franke-Wolfe Algorithm

Find a point $\boldsymbol{v}_k$ by solving a constrained optimization:

$$\boldsymbol{v}_k \in \arg\min_{\boldsymbol{v} \in \mathsf{C}} \langle \boldsymbol{v}, \nabla f(\boldsymbol{x}_k) \rangle. \tag{38}$$



We then set

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \gamma_k(\boldsymbol{v}_k - \boldsymbol{x}_k) = (1-\gamma_k)\boldsymbol{x}_k + \gamma_k \boldsymbol{v}_k \in \mathsf{C}, \tag{39}$$

where $\gamma_k \in (0,1)$ is a specially chosen step size.

Theorem (Convergence of Frank-Wolfe)

Let $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots$ denote the sequence of iterates generated by the Frank-Wolfe method, with step size $\gamma_k = \frac{2}{k+2}$. Then

$$f(\boldsymbol{x}_k) - f(\boldsymbol{x}_\star) \le \frac{2L\mathit{diam}^2(\mathsf{C})}{k+2}. \tag{40}$$

Frank-Wolfe for Matrix Completion

Fact: given a matrix $\boldsymbol{G}$ with SVD $\boldsymbol{G} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^* = \sum_{i=1}^{n_1} \boldsymbol{u}_i \sigma_i \boldsymbol{v}_i$, we have

$$\boldsymbol{V}_\star = -\tau \boldsymbol{u}_1 \boldsymbol{v}_1^* = \arg\min_{\boldsymbol{V}} \langle \boldsymbol{V}, \boldsymbol{G} \rangle \quad \text{subject to} \quad \|\boldsymbol{V}\|_* \le \tau. \tag{41}$$

Frank-Wolfe for Matrix Completion:

1: **Problem:** given $\boldsymbol{Y} = \mathcal{P}_\Omega[\boldsymbol{X}_o + \boldsymbol{Z}] \in \mathbb{R}^{n_1 \times n_2}$ and $\Omega \subseteq [n_1] \times [n_2]$,

$$\min_{\boldsymbol{X}} \frac{1}{2} \|\mathcal{P}_\Omega[\boldsymbol{X}] - \boldsymbol{Y}\|_F^2 \quad \text{subject to} \quad \|\boldsymbol{X}\|_* \le \tau.$$

2: **Input:** $\boldsymbol{X}_0 \in \mathbb{R}^{n_1 \times n_2}$ satisfying $\|\boldsymbol{X}_0\|_* \le \tau$.

3: **for** $(k = 0, 1, 2, \ldots, K-1)$ **do**

4: $\quad (\boldsymbol{u}_1, \sigma_1, \boldsymbol{v}_1) \leftarrow \mathrm{LeadSV}\left(\mathcal{P}_\Omega\left[\boldsymbol{X}_k - \boldsymbol{Y}\right]\right)$ (power iteration).

5: $\quad \boldsymbol{V}_k \leftarrow -\tau \boldsymbol{u}_1 \boldsymbol{v}_1^*$.

6: $\quad \boldsymbol{X}_{k+1} \leftarrow \frac{k}{k+2} \boldsymbol{X}_k + \frac{2}{k+2} \boldsymbol{V}_k$.

7: **end for**

8: **Output:** $\boldsymbol{X}_\star \leftarrow \boldsymbol{X}_K$.

Franke-Wolfe for Noisy Sparse Recovery

Problem: given $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}_o + \boldsymbol{z} \in \mathbb{R}^m$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$,

$$\min_{\boldsymbol{x}} \tfrac{1}{2} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2^2 \quad \text{subject to} \quad \|\boldsymbol{x}\|_1 \le \tau.$$

Input: $\boldsymbol{x}_0 \in \mathbb{R}^n$ satisfying $\|\boldsymbol{x}_0\|_1 \le \tau$.

for $(k = 0, 1, 2, \ldots, K-1)$ **do**

$\quad \boldsymbol{r}_k \leftarrow \boldsymbol{A}\boldsymbol{x}_k - \boldsymbol{y}$.

$\quad i_k \leftarrow \arg\max_i |\boldsymbol{a}_i^* \boldsymbol{r}_k|$ (matching pursuit).

$\quad \sigma \leftarrow \text{sign}\left(\boldsymbol{a}_{i_k}^* \boldsymbol{r}_k\right)$.

$\quad \boldsymbol{v}_k \leftarrow -\tau \sigma \boldsymbol{e}_{i_k}$.

$\quad \boldsymbol{x}_{k+1} \leftarrow \frac{k}{k+2}\boldsymbol{x}_k + \frac{2}{k+2}\boldsymbol{v}_k$.

end for

Output: $\boldsymbol{x}_\star \leftarrow \boldsymbol{x}_K$.

Note: Many greedy variants of the Franke-Wolfe algorithm: Macthing Pursuit (MP), Orthogonal Matching Pursuit (OMP), Compressed Sampling Matching Pursuit (COSAMP), BLITZ, CELER etc.

Other Ideas for Better Scalability

Typical optimization problem: $\min_{\boldsymbol{x}} f(\boldsymbol{x}) = \frac{1}{m}\sum_{i=1}^{m} h_i(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n$.

Complexity $=$ **per iteration cost** $\times \#$ **of iterations**.

- **Block Coordinate Descent** reduces dependency on the dimension n:

$$O(n) \to O(n^{1/2}).$$

- **Stochastic Gradient Descent** (with variance reduction) reduces dependency on sample size m:

$$O(m) \to O(m^{1/2}).$$

- **Acceleration Schemes** reduce the number of iterations k:

$$O(\epsilon^{-2}) \to O(\epsilon^{-1/2}).$$

Assignments

- Reading: Section 8.4 - 8.6 of Chapter 8.
- Programming Homework #3.