EECS 106B February 8, 2022 Lecture

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February 8, 2022

Contents

1 Structure of Nonholominic Systems

We start this lecture discussing about Nonholominic systems and why they are important. This is important because for there are robots that follow the conservation laws and for robots without slipping. Robots without slipping specifically will be important for wheeled robots.

Above mentioned is why learning about Nonholoininc systems are important. For students who previously took EECS 106A, it is important to remember that we actually did not focus on these constraints. We are now focusing on the constraints for this class. The constraints that were mainly covered in EECS106A were mainly algebraic constraints or constraints on the coordinate q.

The first constraint is shown here. It is called the Pfaffian Constraint. I earlier explained how the constraints were algebraic or dependent on the coordinate q. However, Some systems are characterized by having constraints on their velocities.

2 Pfaffian Constraint

For a Pfaffian contraint, the system is characterized by having contraints on the velocity. With this in mind, what is an example of a Pfafffian constraint.

For example, if q $\in R^n$, then a set of constraints in the form

$\omega^i(q)\dot{q} = 0$ for i = 1,...,k

with omega$^{(1)^T}$$(q) \in R^n$ is referred to as a system of Pfaffian constraints. We will assume that the rows $\omega^i(q)$ are linearly independent at q so that the k constraints are independent.

Now the question is can these constraints on the velocity be converted to constraints on the position variables. This brings up the Single Pfaffian Constraint which I will cover next.

3 Single Pfaffian Constraint

The contraints on the velocity can be easily converted to contraints on the position by starting out with a single constraint:

$\omega(q)\dot{q} = \sum_{i=1}^{n} \omega_j(q)\dot{q}_j = 0$

Above is said to be integrable if there exists a function h : $R^n \rightarrow R$ such that

$\omega(q)\dot{q} = 0 \leftrightarrow h(q) = 0$

This implies that there exists some function $\alpha(q)$ called the integrating factor such that

$\alpha(q)\omega(q) = \frac{\partial h}{\partial q_j}(q)$ for j = 1,...,k

We need to find the integrating factor from this. From the equality of the mixed partials of h, that is

$\frac{\partial^2 h}{\partial q_i \partial q_j}(q) = \frac{\partial^2 h}{\partial q_j \partial q_i}(q)$

we get the following

$\frac{\partial(\alpha\omega_j)}{\partial q_i}(q) = \frac{\partial(\alpha\omega_i)}{\partial q_j}(q)$

As simplified as the above equation looks, the problem with this method is that we have to figure out a way to find $\alpha(q)$. This is not the only issue because as the number of constraints grow, you need to not only check the integrability of each constraint but also that of linear combinations of the constraints shown below:

$\sum_{i=1}^{n} \alpha_i(q)\omega^i(q)\dot{q} = 0$

To solve these issues, we will introduce a new Theorem called the Lie (pronounced Lee) Theorem.

4 Equivalent Control Systems

With everything, I just mentioned above, to approach Lie's Theorem, the first order of business will be convert all the constraints on $\dot{q}$ into a control system. Instead of saying that $\dot{q}$ belongs in the nullspace of

$\omega^i(q)$ for $i = 1...k$

we want to say that

$\dot{q}$ belongs in the range of $g_j(q)$ for $j = 1, ..., n - k =: m$

To be more specific, what we are trying to do is construct the right null space of the constraints, denoted $g_j(q)$ for $j = 1, ..., n - k =: m$. That is

$$\omega^i(q) g_j(q) = \begin{cases} 0 & i = 1...k \\ 0 & j = 1, ..., n - k =: m \end{cases}$$

Because of this, the allowable trajectories satisfying the Pfaffian constraints are the trajectories of the control system will be

$\dot{q} = g_1(q)u_1 + ... + g_m(q)u_m$

for suitably chosen inputs $u_1(\cdot), ..., u_m(\cdot), i = 1...m$. This is a drift free control system because if you turn of the controls q stays put/stable.

Next, let's look at some examples of what we just covered.

5 Example 1: Raibert's hopper

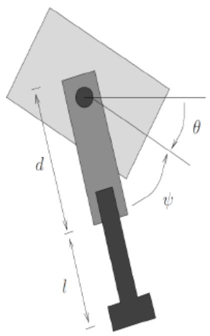

Figure 1: Raibert's hopper

The one legged hopper was originally designed by Marc Raibert to mimic a kangaroo. It has a prismatic joint in the leg and a revolute joint at the hip. The hopping is emulated by the prismatic joint and the swinging of the leg by the hip joint. The lhopper has a stance phase on the ground and a flying phase in the air.

When it is in the air angular momentum is conserved. I is the moment of inertia of the body, the leg mass m is concentrated at the foot. The formula for the angular momentum set to zero is

$$I\dot{\theta} + m(l+d)^2(\dot{\theta} + \dot{\psi}) = (I + m(l+d)^2)\dot{\theta} + m(l+d)^2\dot{\psi} = 0$$

If $q = (\Psi, \text{I}, \theta)^T$ then an equivalent control system for describing it is found by finding a bass for the null space of

An especially convenient one is

Now let's look at the next example.

$$\omega^1(q) = \begin{bmatrix} m(l+d)^2 & 0 & I+m(l+d)^2 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} 1 \\ 0 \\ -\frac{m(l+d)^2}{I+m(l+d)^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2$$

6 Example 2: Planar Space Robot

Below is a simplified model of a robot in space with two arms connected to the body through revolute joints.

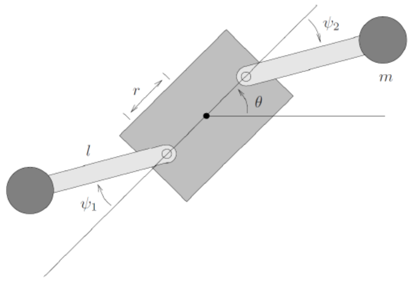

Figure 2: Planar Space Robot

The mass and moment of inertia of the central body are M; I and the mass of each arm is m concentrated at the ends of the arms of length l.

For the diagram above, the statement of angular conservation is shown below. Note that the Lagrangian does not depend on the body angle θ.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} = 0 = a_{13}(\psi)\dot{\psi}_1 + a_{23}(\psi)\dot{\psi}_2 + a_{33}(\psi)\dot{\theta}$$

Setting $q = (\Psi_1, \Psi_2, \theta)^T$ we get the equivalent control system

$$\dot{q} = \begin{bmatrix} 1 \\ 0 \\ -\frac{a_{13}}{a_{33}} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ -\frac{a_{23}}{a_{33}} \end{bmatrix} u_2$$

In MLS, page 335 there is a detailed derivation of the Lagrangian equations shown above for the Space Robot.

Now, let's look at the next example.

7 Example 3: Rolling without slipping

A second source of nonholonomy is from constraints that arise from discs, wheels which roll without slipping. Consider a penny rolling on a surface:

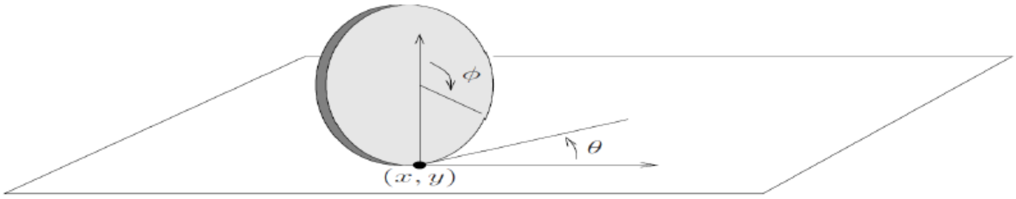

Figure 3: Rolling Without Slipping

Here x, y are the location of the contact point on the plane θ is the angle that the disk makes with the horizontal, ϕ is the angle made by a fixed line on the disk relative to the vertical axis. ρ is the radius of the disk.
Before getting into any calculations, what exactly does rolling without slipping mean. Rolling Without Slipping means the velocity perpendicular to the direction which the penny is pointing towards is zero. The way you compute this is now shown below.

If the disk roles without slipping we have with $q =$ (x,y,$\theta, \phi)^T \in R^4$

$$\begin{aligned} \dot{x} - \rho\cos\theta\dot{\phi} &= 0 \\ \dot{y} - \rho\sin\theta\dot{\phi} &= 0 \end{aligned}$$

This may be written as

$$\begin{bmatrix} 1 & 0 & 0 & -\rho\cos\theta \\ 0 & 1 & 0 & -\rho\sin\theta \end{bmatrix} \dot{q} = 0$$

Thus, there are 2 Pfaffian constraints on R^4. A convenient choice of control system, with $\dot{\theta} = u_1$ and $\dot{\phi} = u_1$ is

$$\dot{q} = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2$$

This is a two input control system.

8 Example 4: Front Wheel Drive Car

Below is a picture of a front wheel drive car. The steering angle is ϕ, the angle of the car body is θ and the position of the midpoint of the rear axle is x, y.

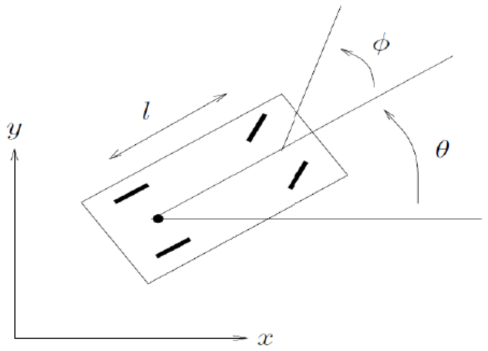

Figure 4: Front Wheel Drive Car

This is sometimes referred to as the kinematic model of a car. It is used frequently in the analysis of self driving cars and their motion plans. Let's dive into more detail with this model.

The rolling without slipping constraints for the front wheels and back wheels are a statement that the velocity perpendicular to the direction that the velocity of the wheels perpendicular to the direction they are pointing is 0:

$$\sin(\theta + \phi)\dot{x} - \cos(\theta + \phi)\dot{y} - l\cos\phi\,\dot{\theta} = 0$$
$$\sin\theta\,\dot{x} - \cos\theta\,\dot{y} = 0$$

Using the steering velocity as $u_2 = \phi$ and $q = (x, y, \theta, \phi) \in R^4$ gives the control system

$$\dot{q} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{l}\tan\phi \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

u_1 has the interpretation of the driving input and u_2 as the steering input.

Now let's look at the next example.

9 Example 5: Car with N Trailers

The figure shows a car with N trailers attached. The hitch of each trailer is attached to the center of the rear axle of the previous trailer. The wheels of the individual trailers are aligned with the body of the trailer.

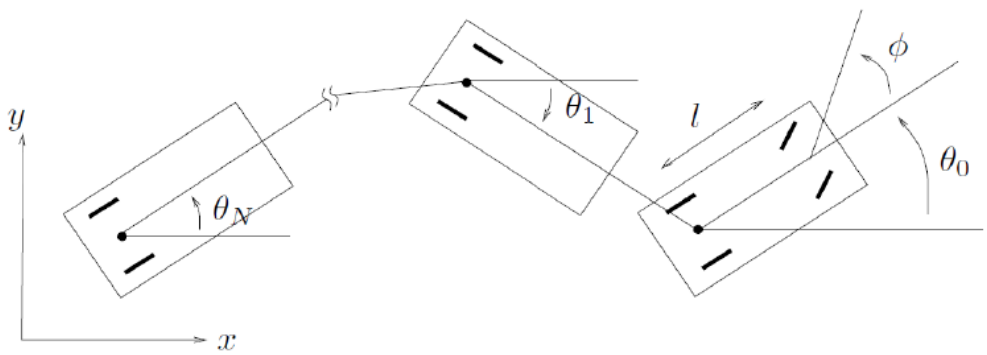

Figure 5: Car With N Trailers

Here we can see that $q = (x, y, \phi, \theta_0, .., \theta_n)^T \in R^{N+4}$. There are $N + 2$ sets of wheels which roll without slipping to give $N + 2$ Pfaffian constraints.

For a more detailed explanation of what is going navigate to Exercise 6 in Chapter 7 of MLS.

10 Example 6: A Firetruck

The figure shows a kinematic model of a fire truck. You may have noticed that there is a driver in the front and one more at the back of the ladder. It is not unlike a car with one trailer, except that the rear axle is also steerable.

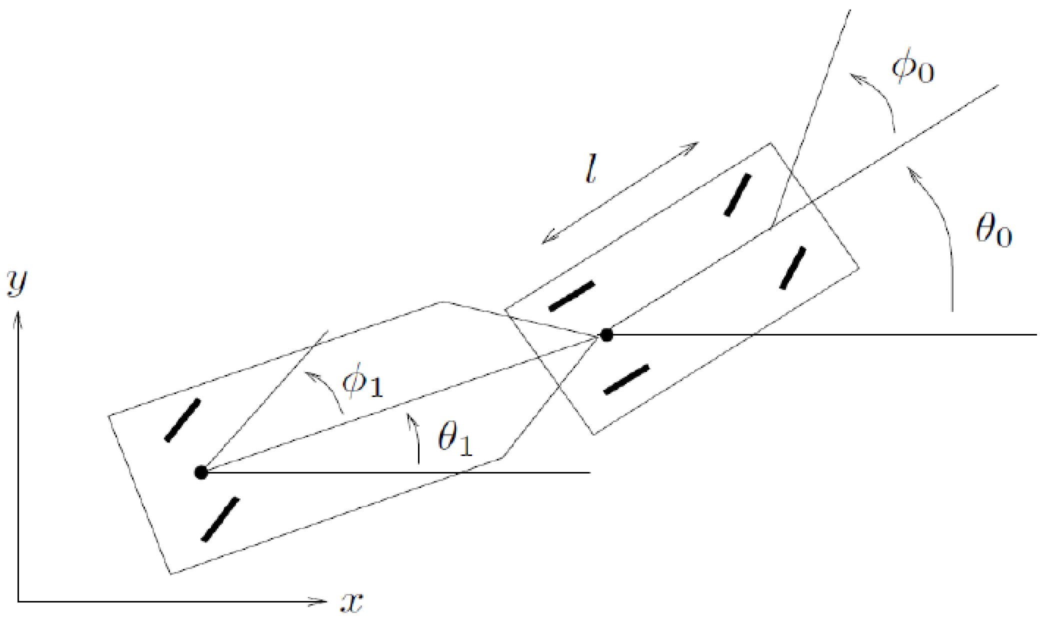

Figure 6: Firetruck

Navigate to Exercise 7 in Chapter 7 of MLS for more details about this diagram.

11 Definition of Holonomic Systems

Before discussing about controlability, let's discuss about the Definitions of Holonomic Systems.

The set of Pfaffian constraints $\omega^i(q)$ for $i = 1, ..., k$ is said to be holonomic if there exists functions $h^i(q)$ for $i = 1, ..., k$ such that

$$\omega^i(q)\dot{q} = 0 \leftrightarrow h^i(q) = c_i, \ i = 1, ...k$$

That is the number of constraints on q are precisely k and thus q lies on a manifold of dimension $(n - k)$. On the other hand if there are only $p < k$ functions such that

$$\omega^i(q)\dot{q} = 0 \leftrightarrow h^i(q) = c_i, \ i = 1, ...p$$

the Pfaffian system is said to be nonholonomic. If $p = 0$ the Pfaffian system is said to completely nonholonomic. For nonholonomic systems there are fewer than k contraints on the state space q. For completeley nonholonomic systems there are NO constraints on q. If $0 < p < k$ the constraints are called partially nonholoomic.

Now, let's move onto controlability.

12 Lie Bracket and Lie Product

The Lie bracket encode the states where a non-holomic system could potentially get to. Take for example the system with control system

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

This shows that this non-holonomic system can only move in two direction, g_1 and g_2. However, Lie showed that for a small amount of time ϵ, by going in direction g_1 and g_2 and then $-g_1$ and $-g_2$, each for ϵ amount of time, there is a net movement.

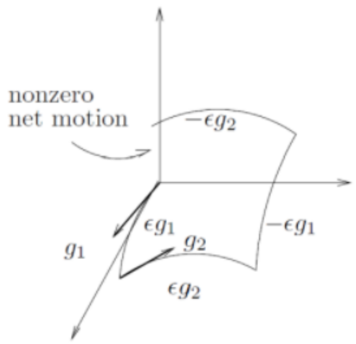

Figure 7: Net Motion

A small amount of movement is a derivative, and hence by Taylor expansion, the Lie bracket is defined as

$$[g_1, g_2](q) = \frac{\partial g_2}{\partial q} g_1(q) - \frac{\partial g_1}{\partial q} g_2(q)$$

A real life example of this could be found in the action of parallel parking. Normally, shifting over a car horizontally is impossible to do with just a horizontal force, however following the steps outlined in the Lie product, steer right move forward, steer left move backwards, the car achieved the parallel net movement. This intuitive show how the Lie bracket enclose states the non-holonomic system could achieve.
Lie products are just a series of Lie bracket such as this one:
$[[f, g], [f, [f, g]]]$

Lie products also have a nice set of properties including:

1. $[f, g] = -[g, f]$ — Skewed Symmetry
2. $[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0$ — Jacobi Identity
3. $[\alpha f, \beta g] = \alpha\beta[f, g] + \alpha(L_f\beta)g - \beta(L_g\alpha)f$ — Chain Rule

13 Distributions and Involutivity

A distribution Δ is defined as:

$$\Delta(q) = span(g_1(q), ..., g_m(q))$$

Which essentially means all the possible directions the system can go in, including the ones denoted by the Lie product, so for example in a car, forward and backward, steering left and right, and parallel movement left and right are all part of the distribution.
The distribution is called **regular** if the dimension of the distribution does not change with q, in other words, directions the system can reach does not depend on the system's position. The distribution is **involutive** if the set of directions doesn't grow no matter what order the brackets are in.

$$\forall f, g \in \Delta, [f, g] \in \Delta$$

14 Frobenius Theorem

A distribution is **integrable** if there exists functions h, that for all the directions f in the distribution, the product of the partial of the function and the direction in the distribution is 0.

$\forall f \in \Delta, L_f h_i = \frac{\partial h_i}{\partial q} f(q) = L_f h_i(q) = 0$

The **integral manifolds of** Δ are then the set of all values of h evaluated on q.

$M_c = h_1(q) = c_1, ..., h_{n_k}(q) = c_{n-k}$

Frobenius Theorem states that a regular distribution is integrable iff the distribution is involutive. This implies that the more derivative you take, or the Lie Brackets you make, the more places you can go and the less constraints there are.

15 Controllability and Chow's Theorem

The places a system can reach is defined by its reach set, or the positions the system can be steered from an initial point in t seconds while staying inside the set of possible initial positions.

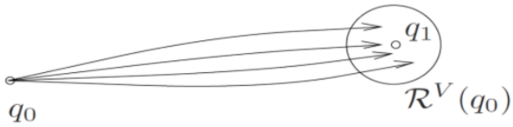

Figure 8: Reachable Set

Chow's theorem suggest that if you take all the possible Lie Bracket and the span of those bracket is in the full dimension of space, then you can reach the neighborhood of the origin. More formally:

If $\bar{\Delta}(q) = \Re^n$ *for all* q *in a neighborhood of* q_0*, then* $\mathscr{R}^V(q_0, \leqslant T)$ *has non empty interior.*

Figure 9: Chows

16 Nonholonomic Integrator

The nonholonomic integrator for our state space q is as follows:

$$\dot{q_1} = u_1$$

$$\dot{q_2} = u_2$$

$$\dot{q_3} = q_1 u_2 - q_2 u_1$$

The system with the dt dropped can be written as:

$$dq_3 = q_1 dq_2 - q_2 dq_1$$

16.1 Optimal Control

Brockett showed that the optimal input to minimize the cost function

$$\int_0^1 ||u(t)||^2 \, dt$$

is a sinusoidal function

The generalization of this system with >2 inputs is the following where q ϵ R^m and Y ϵ so(m)

$$\dot{q} = u$$
$$\dot{Y} = qu^T - uq^T$$

17 Chained Form Systems

17.1 1-Chain Form

The nonholonomic integrator extended to n-dimensions is known as Goursat normal form systems

17.1.1 Controllability of 1-Chain System

By way of notation we define

$$ad_{g1}g_2 = [g_1, g_2]ad_{g1}^{k+1} = [g_1, ad_{g1}^{k}g_2]$$

17.2 Rectification using sinusoids

First steer q_1 and q_2. Then use $u_i(t) = asin2\pi t$ and $u_2(t) = bcos2\pi t$ to steer z_3. After one second $q_1(1) = q_1(0), q_2(1) = q_2(0)$, $q_3(t) = q_3(0) + \frac{1}{2}\frac{ab}{\pi}(t - \frac{\sin 2.2\pi t}{2.2\pi})$. The constant term in the integrand is what allows rectification and after one second q_3 increases.

$$q_3(1) = q_3(0) + \frac{1}{2}\frac{ab}{\pi}(t - \frac{\sin 2.2\pi t}{2.2\pi})$$