EECS C106B / 206B Robotic Manipulation and Interaction **Spring 2020**

Lecture 6: (Quadrotors and Nonholonomic Constraints)

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6.1 Quadrotors

6.1.1 Fundamental questions

How much power to hover? Model air as incompressible fluid and analyze velocity and pressure gradients.

Conservation of mass

$$\rho A v_i = \rho A_2 v_2$$

Conservation of momentum

$$T = (\rho A v_i) v_2$$

(mass flow rate x change in velocity)

Thrust of Pressure Difference

$$T = A(p_L - p_U)$$

Induced Power/Ideal Power

$$P_{induced} = \frac{T^{3/2}}{\sqrt{2\rho A}}$$

Propellor "Figure of Merit": Fraction of aerodynamic shaft power converted to useful aerodynamic induced power. Typically 0.3-0.6.

$$FM = \frac{P_{induced}}{P_{shaft}}$$

Ground Effect Induced power is lower when hovering near the ground.

6.1.2 Demonstrated experiments

Human can produce enough to hover! (npr video)

Gyroplanes going back to 1907.

Coordinated assembly, moving rotors, morphology. Maneuverability. Planes + quadrotors.

6.1.3 Equations of control

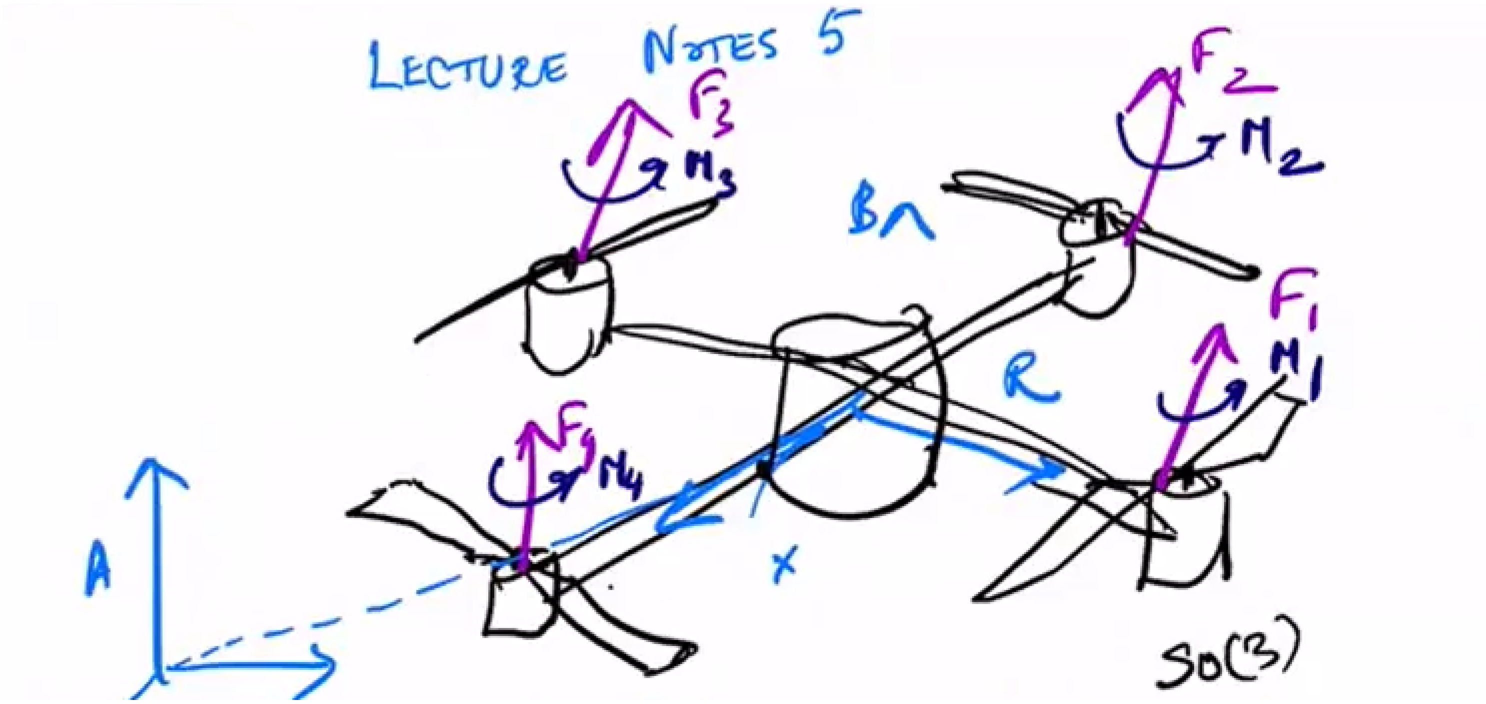


Figure 6.1: Quadrotor Diagram

$$F_i = k_F \sigma_i^2$$

$$M_i = k_M \sigma_i^2$$

$$m\ddot{x} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$$

$$\dot{R} = R\hat{\omega}$$

$$I\dot{\omega} + \omega \times I\omega = \begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & L & 0 & -L \\ -L & 0 & L & 0 \\ \gamma & -\gamma & \gamma & -\gamma \end{bmatrix} \begin{bmatrix} k_P \sigma_1^2 \\ k_P \sigma_2^2 \\ k_P \sigma_3^2 \\ k_P \sigma_4^2 \end{bmatrix}$$

Sigmas controlled by motor servos

Abstracted away we get

$$
\begin{aligned}
m\ddot{x} &= \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix} \\
\dot{R} &= R\hat{\omega} \\
I\dot{\omega} &= -\omega \times I\omega + \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} \\
R &= e^{\hat{z}\psi} e^{\hat{y}\theta} e^{\hat{x}\phi}
\end{aligned}
$$

(yaw, pitch, roll respectively)

$$\hat{\omega} = R^T \dot{R} = \hat{z}\dot{\psi} + e^{-\hat{z}\psi}\hat{y}e^{\hat{z}\psi}\dot{\theta} + e^{-\hat{z}\psi}e^{-\hat{y}\theta}\hat{x}e^{\hat{y}\theta}e^{\hat{z}\psi}$$

$$\begin{bmatrix} \dot{\gamma} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = J(\gamma, \theta, \phi)\omega$$

Call roll, pitch and yaw the outputs $\mathbf{y}$.

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix}$$

$$\dot{y_4} = \dot{\gamma} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} J\omega$$

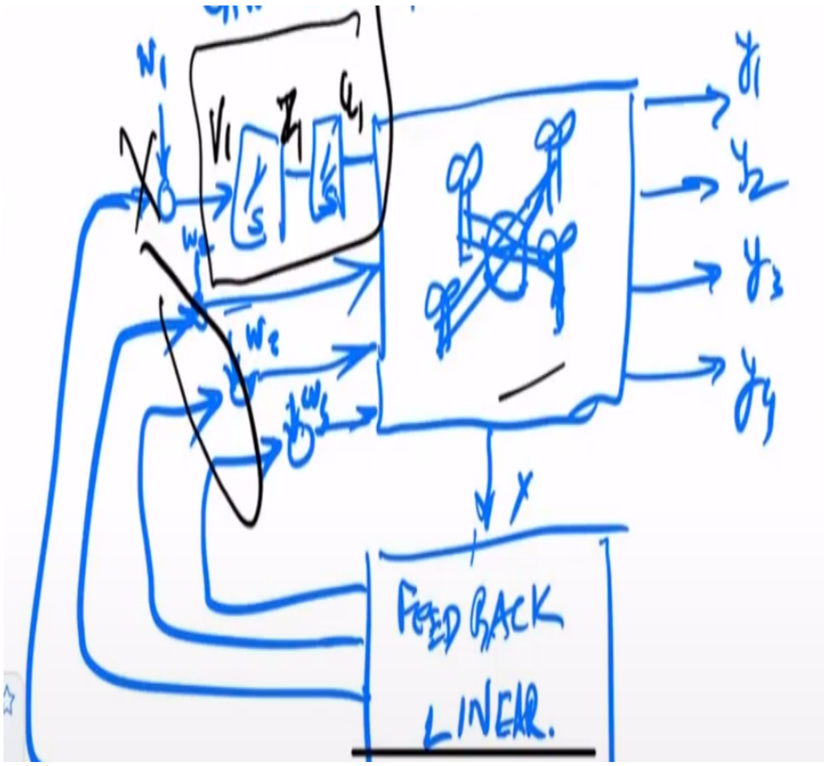
Differentiate until the inputs show up

$$\ddot{y}_4 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \dot{J}\omega + \begin{bmatrix} R_z & & 0 & \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Rank 2 matrix! so we can't take the inverse to solve for the input.

Dynamic Extension We keep differentiating the dynamics until the input term is nonzero, then keep differentiating until it has a nonsingular matrix.

On the third derivative of y, we get a rank 4 matrix if $u_1 \neq 0$ and $a_{44} \neq 0$. Can linearize and decouple.



6.2 Nonholonomic Systems

Examples

- twirling a pencil around your fingers! Making and braking contact.
- rolling with the constraint of not slipping
- how to reorient yourself in space

Nonoholonomic mechanics.

6.2.1 Pfaffian Constraints

Constraints on the velocities. Given state $q \in \mathcal{R}^n$, constraints $i = 1, ..., k$ of the form

$$\omega^i(q)\dot{q} = 0$$

with $\omega^i(q) \in \mathcal{R}^n$ are Pfaffian. Assume the rows are linearly indpeendent at q so that the constraints are linearly independent.

Given such constraints, can we convert them into constraints on the states instead?

A single constraint is said to be integrable if $\exists h : \mathcal{R}^n \to \mathcal{R}$ s.t.

$$\omega^i(q)\dot{q} = 0 \iff h(q) = 0$$

From

$$\frac{\partial^2 h}{\partial q_i \partial q_j} = \frac{\partial^2 h}{\partial q_j \partial q_i}$$

we get

$$\frac{\partial(\alpha w_j}{\partial q_i} = \frac{\partial(\alpha w_i}{\partial q_j}$$

A set of Pfaffian constraints is **holonomic** if there exists $h_i(q)$ for $i = 1, \ldots, k$ such that

$$\omega^i(q)\dot{q} = 0 \Leftrightarrow h_i(q) = c$$

It is **nonholonomic** if there are $p < k$ such constraints, **partially nonholonomic** if $p > 0$ and **completely nonholonomic** if $p = 0$ (there are no such constraints).

Constraints in velocity appearing as constraints in the state. Limiting the state to a $(n - p)$ dimensional manifold.

To get the directions we can move we construct the right null space of the constraints. That is

$$w^i(q)g_j(q) = 0$$

The allowable trajectories satisfying the Pfaffian constraints are the trajectories of the control system.

$$\dot{q} = g_1(q)u_1 + \cdots + g_m(q)u_m$$

6.2.2 Examples

6.2.2.1 Raibert's hopper

2 DoF: 1 rotates leg, and 1 extends and retracts.
What is the control law required to make this robot flip in the air?

Pfaffian constraint (angular momentum is conserved)

$$I\dot{\theta} + m(I + d)^2(\dot{\theta} + \dot{\psi}) = (I + m(I + d)^2)\dot{\theta} + m(I + d)^2\dot{\psi} = 0$$

$$\dot{q} = \begin{bmatrix} 1 \\ 0 \\ -\frac{m(I+d)^2}{I+m(I+d)^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2$$

6.2.2.2 Planar space robot

Reorient satteliates without using boosters.

Angular momentum constraints on Lagrangian leads to state $q = (\psi_1, \psi_2, \theta)^T$ having the control system

$$\dot{q} = \begin{bmatrix} 1 \\ 0 \\ -\frac{a_{13}}{a_{33}} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ -\frac{a_{23}}{a_{33}} \end{bmatrix} u_2$$

6.2.2.3 Rolling without slipping

State $q = (x, y, \theta, \phi)^T$

$$\dot{x} - \rho \cos\theta \dot{\phi} = 0$$
$$\dot{y} - \rho \sin\theta \dot{\phi} = 0$$

Control law

$$\dot{q} = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2$$

6.2.2.4 Front Wheel Drive Car

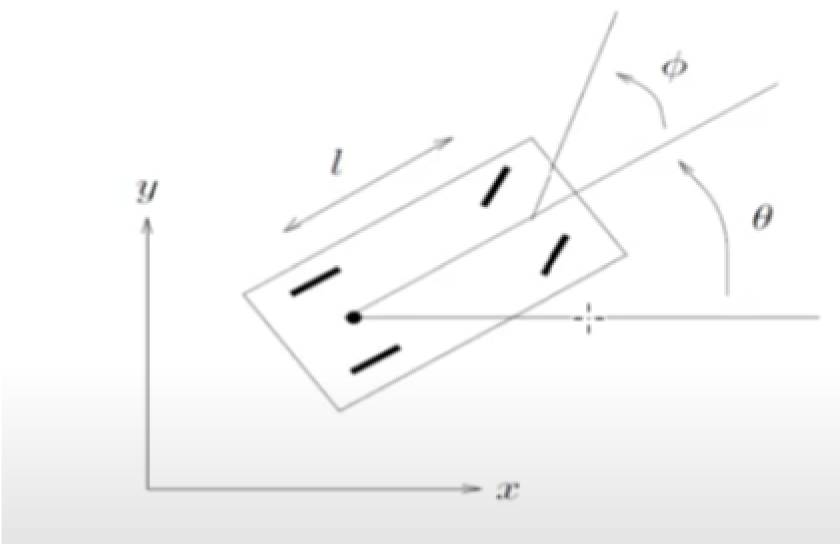


Figure 6.2: Front Wheel Drive Car

Kinematic model of a car. Steering angle ϕ, angle of car body is θ, position x, y.

Constraints

$$\sin(\theta + \phi)\dot{x} - \cos(\theta + \phi)\dot{y} - I\cos(\phi)\dot{\theta} = 0$$
$$\sin(\theta)\dot{x} - \cos(\theta)\dot{y} = 0$$

Control law

$$\dot{q} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{l}\tan\phi \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

6.2.2.5 Car with N trailers

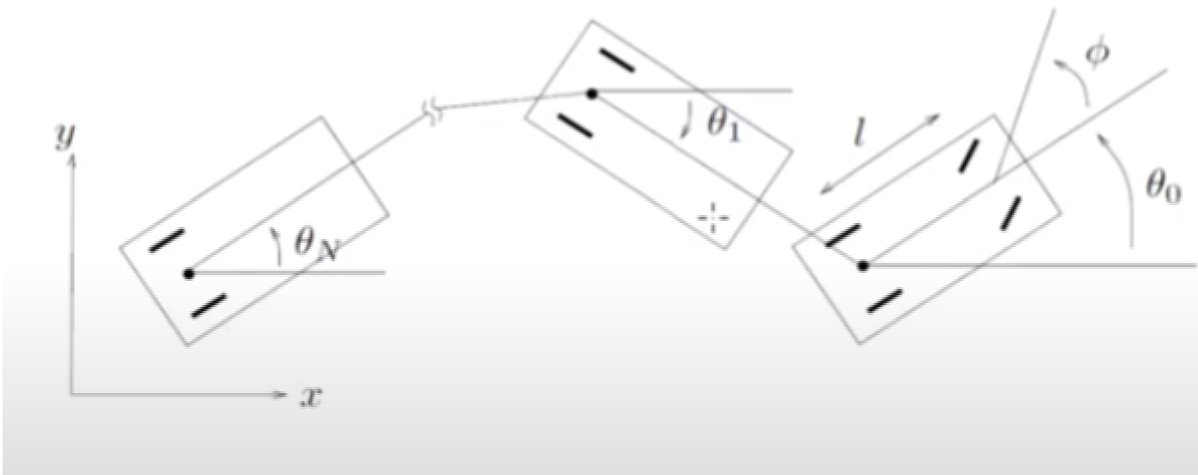


Figure 6.3: Car with N Trailers

$$q = (x, y, \phi, \theta_0, \ldots, \theta_N)^T \in \mathcal{R}^{N+4}$$

$N + 2$ sets of wheels which roll without slipping gives $N + 2$ Pfaffian constraints.

6.2.2.6 Firetruck

Rear axle is also steerable (driver in the front and driver in the back). Velocity tangential to the wheels is 0.