EECS C106B / 206B Robotic Manipulation and Interaction **Spring 2022**

Lecture 4: (Nonlinear Control Cont.)

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4.1 Review

Given a S.I.S.O. function where

$$\dot{x} = f(x) + g(x)u \qquad x \in \mathbb{R} \quad u \in \mathbb{R}$$

$$y = h(x) \qquad y \in \mathbb{R}$$

$$\frac{d}{dt}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_n\end{bmatrix} = \begin{bmatrix}f_1(x_1...x_n)\\ f_2(x_1...x_n)\\ \vdots\\ f_n(x_1...x_n)\end{bmatrix} + \begin{bmatrix}g_1(x_1...x_n)\\ g_2(x_1...x_n)\\ \vdots\\ g_n(x_1...x_n)\end{bmatrix}u \tag{4.1}$$

$$\dot{y} = \frac{d}{dt}y(k) = \frac{d}{dt}h(x(t)) = \frac{\partial}{dx}h(x)\cdot\dot{x} = Dh(x)[f(x)+g(x)u] = L_f(x) + L_ghh(x)u$$

$L_fh(x)$ is the Lie Derivative. g,h in the direction f

$$\text{If} \quad L_gh(x) \neq 0 \longrightarrow M = \frac{1}{L_gh(x)}[-L_fh(x) + v]$$

4.1.1 Input Output Linearization f

$$\dot{y} = v \quad v = \dot{y_{des}} + k_1(y_{des} - y)$$

$$u = \frac{1}{L_gh(x)}[-L_fh(x) + \dot{y_{des}} + k_1(y_{des} - h(x))]$$

4.1.2 Nonlinear Control Law

$$u = \frac{1}{L_gh(x)}[-L_fh(x) + \dot{y_{des}} + k_1(y_{des} - h(x))]$$

$$\longrightarrow \dot{y} = \dot{y_{des}} + k_1(y_{des} - y)$$

$$y_{des} - y = e \longrightarrow \dot{e} + k_1e = 0$$

$$\text{If} \quad L_gh(x) = 0\dot{y} = L_fh(x) + L_gh(x)u = L_fh(x)$$

$$\begin{aligned}\ddot{y} &= \frac{d}{dt}(L_f h(x)) \\ &= \frac{d}{dx}(L_f h(x))(f(x)+g(x)u) \\ &= L_f(L_f h) + L_f(L_g h)u \\ &= L_f \circ L_f h + L_f \circ L_g h u \\ &= L_f^2 h(x) + L_f L_g h\end{aligned} \tag{4.2}$$

$$\text{If} \quad L_f L_g h(x) \neq 0$$

$$u = \frac{1}{L_f L_g h(x)}[-L_f^2 h(x) + v] \quad \ddot{y} = v \quad y \longrightarrow y_{des}$$

$$\begin{aligned}v &= y_d\ddot{e}s + k_1(\dot{y_{des}} - \dot{y}) + k_2(y_{des} - y) \\ &= \dot{y_{des}} + k_1(\dot{y_{des}} - L_f h(x)) + k_2(y_{des} - h(x))\end{aligned} \tag{4.3}$$

$$\ddot{e} + k_1\dot{e} + k_2 e = 0 \quad e = y_{des} - y$$

4.1.3 Full Generalization

Let r be the lowest integer such that

$$L_g h(x) = 0 = L_g L_f h(x) = ... = L_g L_f^{r-2} h(x) \quad L_g L_f^{r-1} h(x) \neq 0$$

then

$$\begin{aligned}y &= h(x) \\ \dot{y} &= L_f h(x) + L_g h(x) u \\ \ddot{y} &= L_f^2 h(x) + L_g L_f h(x) u \\ y^{(x)} &= L_f^r h(x) + L_g L_f^{r-1} h(x) u\end{aligned}$$

r is the number of terms needed to differentiate y until the ?? appears on the right hand side

4.2 Two Input Two Output

Now, consider the two input, two output (TITO) case.

$$\begin{aligned}\dot{x} &= f(x) + g_1(x)u_1 + g2(x)u_2 \\ y_1 &= h_1(x)y_2 &&= h_2(x)\end{aligned}$$

Here, we'll follow the same procedure as before, with the only difference being that we'll proceed one output at a time. We begin:

$$\begin{aligned}\dot{y}_1 &= \frac{d}{dt}h_1(x) \\ &= \frac{d}{dx}h_1(x)[f(x) + g_1(x)u_1 + g_2(x)u_2] \\ &= L_f h_1 + L_{g_1} h_1 u_1 + L_{g_2} h_1 u_2\end{aligned}$$

If either $L_{g_1}h_1u_1$ or $L_{g_2}h_1u_2$ is nonzero, we stop. Otherwise, we differentiate again:

$$\ddot{y}_1 = L_f^2h_1 + L_{g_1}L_fh_1u_1 + L_{g_2}L_fh_1u_2$$

Let r_1 be the first deriative at which we get a nonzero term. Then:

$$y_1^{r_1} = L_f^{r_1}h_1 + L_{g_1}L_f^{r_1-1}h_1u_1 + L_{g_2}L_f^{r_1-1}u_1$$

Applying the same concept for r_2, we write:

$$y_2^{r_2} = L_f^{r_2}h_2 + L_{g_1}L_f^{r_2-1}h_2u_1 + L_{g_2}L_f^{r_2-1}u_2$$

We may now rearrange the above in matrix form as:

$$\begin{bmatrix} y^{r_1} \\ y^{r_2} \end{bmatrix} = \begin{bmatrix} L_f^{r_1}h_1 \\ L_f^{r_2}h_2 \end{bmatrix} + \begin{bmatrix} L_{g_1}L_f^{r_1-1}h_1 & L_{g_2}L_f^{r_1-1}h_1 \\ L_{g_1}L_f^{r_2-1}h_2 & L_{g_2}L_f^{r_2-1}h_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The 2x2 matrix in the above equation is commonly referred to as $A(x)$. If $A(x)$ is invertible, consider the following formulation for the system inputs:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -A^{-1}(x)\Big(\begin{bmatrix} L_f^{r_1} & h_1 \\ L_f^{r_2} & h_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Big)$$

Plugging into the output equation:

$$\begin{bmatrix} y_1^{r_1} \\ y_2^{r_2} \end{bmatrix} = \begin{bmatrix} L_f^{r_1}h_1 \\ L_f^{r_2}h_2 \end{bmatrix} + A(x)u$$

This then yield the following linearized system:

$$\begin{bmatrix} y_1^{r_1} \\ y_2^{r_2} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Where v_1 and v_2 are traditionally designed inputs. As can be seen in the above, we've taken a TITO non-linear system and transformed it into a system that's *both* linear and decoupled! Wow!
As a brief review of vocabulary, recall that y_1 and y_2 are said to have "relative degrees" of r_1 and r_2, respectively.
What can we conclude from this result? We know that if A^-1 exists, $(det(A(x) \neq 0)$, then we can both linearize and decouple the system's dynamics. What happens, however, if A is not invertible? We must use a clever "hack" - a *dynamic extension.* Let's illustrate how we may apply this process with the following example: a planar quadrotor.

4.2.1 Dynamic Extension of the Planar Quadrotor

Recall the following dynamics model for the planar quadrotor:

$$\begin{bmatrix} y \\ z \\ \phi \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{x_5} \\ \dot{x_6} \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-1}{m}\sin(x_3) \\ \frac{1}{m}\cos(x_3) \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{I} \end{bmatrix} u_2$$

$$y_1 = x_1$$
$$y_2 = x_2$$

Once again, we must follow the procedure of differentiating the outputs, y_1 and y_2, until the system inputs appear. This will allow us to begin the process of feedback linearization. Let's begin:

$$\dot{y_1} = \dot{x_1} = x_4$$
$$\dot{y_2} = \dot{x_2} = x_5$$

No luck yet - the inputs still haven't appeared! Let's continue differentiating:

$$\ddot{y}_1 = \dot{x_4} = \tfrac{-1}{m}\sin(x_3)u_1$$
$$\ddot{y}_2 = \dot{x_5} = -g + \tfrac{1}{m}\cos(x_3)u_1$$

Thus, we find that the relative degrees, r_1 and r_2, are both 2. Let's proceed to put this into the TITO form from our previous derivation:

$$\begin{bmatrix}\ddot{y}_1\\ \ddot{y}_2\end{bmatrix} = \begin{bmatrix}0\\ -g\end{bmatrix} + \begin{bmatrix}\frac{-1}{m}\sin(x_3) & 0\\ \frac{1}{m}\cos(x_3) & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

The above system *is* decoupled, but has a problem - due to the A matrix having a column of zeros, we conclude that the A matrix is *singular* and thus noninvertible. This is because at the moment, u_2 has no effect on the system. Upon inspecting our dynamics equations, we find that we can only "get" to u_2 from $\dot{x}_6$.
To introduce u_2 into the system, we use the following "hack." Define a new state, z_1, as follows:

$$u_1 = z_1$$
$$\dot{z_1} = v_1$$

Where v_1 is a new input to the system. What's really happening here is the following - we're introducing an integrator into the system. By integrating v_1, we get u_1, which is our normal input to the system.
Thus, we increase the state dimension by 1 and get the system:

$$\begin{bmatrix}\dot{x}\\ \dot{z_1}\end{bmatrix} = \begin{bmatrix}f(x) + g_1(x)z_1\\ 0\end{bmatrix} + \begin{bmatrix}g_2(x)\\ 0\end{bmatrix}u_2 + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}v_1$$
$$\ddot{y}_1 = \frac{-1}{m}\sin(x_3)u_1$$
$$\ddot{y}_2 = -g + \frac{1}{m}\cos(x_3)u_1$$

Now, let's continue to differentiate y_2:

$$\dddot{y}_2 = \frac{-1}{m}\sin(x_3)z_1x_6 + \frac{1}{m}\sin(x_3)v_1$$

Now, define another new state, $v_1 = z_2$, where $\dot{z_2} = w$, and w is another input.

$$y_1^{(4)} = \frac{-1}{m}\sin x_3x_6^2z_1 - \frac{1}{m}\cos x_3z_2x_6 - \frac{1}{m}\cos x_3z_1u_2 - \frac{1}{m}\cos x_3x_6v_1 - \frac{1}{m}\sin x_3w_1$$

Now, we see that u_2, as well as our new inputs, are all in the system equations! Let's calculate the next derivative of y_2:

$$y_2^{(4)} = \frac{-1}{m}\cos x_3x_6^2z_1 - \frac{1}{m}\sin x_3z_2x_6 - \frac{1}{m}\sin x_3z_1u_2 + \frac{1}{m}\cos x_3x_6v_1 + \frac{1}{m}\sin x_3w_1$$

Now, we get u_2 and other inputs appearing in the equation for u_2! Let's write out the fully realized equation in matrix form. Note that the first vector, known as the drift vector, has been omitted due to its excessive complexity:

$$\begin{bmatrix} u_1^{(4)} \\ u_2^{(4)} \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} + \begin{bmatrix} \frac{-1}{m}\cos x_3 & \frac{1}{mI}\sin x_3 z_1 \\ \frac{1}{m}\sin x_3 & \frac{-1}{mI}\cos x_3 \end{bmatrix} \begin{bmatrix} w_1 \\ u_2 \end{bmatrix}$$

Now, our A matrix is invertible for the following condition:

$$det(A) = \frac{1}{m^2 I} z_1 \neq 0$$

Thus, as long as $z_1 \neq 0$, the A matrix is invertible. Assuming this condition, we may now proceed as standard in our linearization problems. We may choose w_1, u_2 such that the system's nonlinearities are canceled out, leaving only a set of linear error differential equations behind.

The technique used to to achieve this method for noninvertible A is known as *dynamic extension*. An extremely large class of nonlinear systems may be converted to this form. Recall the main idea: by defining new states via integrators, we may introduce the input terms into our dynamics equations.