C106B - Robotic Manipulation and Interaction **(Week 4)**

Discussion #4

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Problem 1 - Pfaffian Constraints

When we study nonholonomic systems, we are concerned with velocity constraints, or constraints on the derivative of our system state. You can represent a velocity constraint as the level set of any function incorporating state and velocity $h(q, \dot{q}) = 0$, but we are specifically interested in the subset of velocity constraints that can be represented as

$$A(q)\dot{q} = 0$$

A is a $k \times n$ matrix, where n is the number of states, and k is the number of constraints. This kind of constraint is called a Pfaffian constraint.

MLS uses two notations to describe Pfaffian constraints. It describes a set of Pfaffian constraints as $A(q)\dot{q} = 0$, and a singular constraint as $w(q)\dot{q} = 0$. In this case,

$$A = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix}$$

1. Let's reconsider the Raibert hopper discussed in lecture

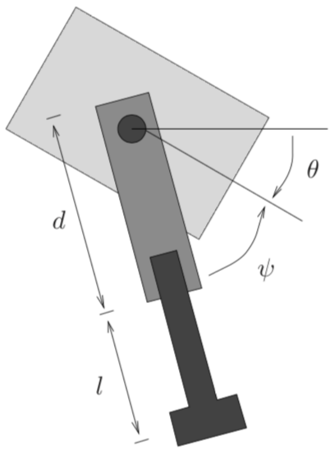

Figure 1: A simplified model of a Raibert Hopper

This system has three state variables $q = (\phi, l, \theta)$, and two inputs $\dot{\phi}$ and $\dot{l}$. The constraint on this system is conservation of angular momentum, the initial value of which we assume is zero.

$$I\dot{\theta} + m(l+d)^2(\dot{\theta} + \dot{\phi}) = 0$$

(a) Express the conservation of angular momentum constraint as a Pfaffian constraint $A(q)\dot{q} = 0$.
We can rewrite the given formula in the form of

$$\begin{bmatrix} m(l+d)^2 & 0 & I + m(l+d)^2 \end{bmatrix} = A(q)\dot{q} \quad (0.1)$$

(b) Setting our system inputs as $u_1 = \dot{\phi}$ and $u_2 = \dot{l}$, express the dynamics of the system in the form

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

Where $g_1(q)$ and $g_2(q)$ form a basis for the null space of $A(q)$
Given that we have

$$A(q) = \begin{bmatrix} m(l+d)^2 & 0 & I + m(l+d)^2 \end{bmatrix} \tag{0.2}$$

by inspection, we can pull out the null space as

$$\dot{q} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -\frac{m(l+d)^2}{I+m(l+d)^2} \end{bmatrix}}_{g_1(q)} u_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{g_2(q)} u_2 \tag{0.3}$$

enumerate

(c) Now let's examine the a Dubins car, or unicycle model robot.

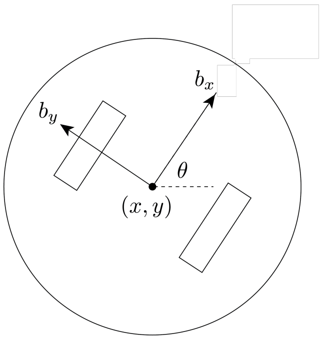

Figure 2: A unicycle model robot

This system has three state variables $q = (x, y, \theta)$, and two control inputs $v = \dot{x}_b$ and $\omega = \dot{\theta}$. The constraint on this system is that sideways motion $\dot{y}_b$ is zero.

(a) Express the sideways motion constraint as a Pfaffian constraint in terms of the state variables q: $A(q)\dot{q} = 0$.
The sideways motion constraint tells us that the robot has no component of its linear velocity in the direction of b_y. We can write b_y in terms of the robot's heading as

$$b_y = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \tag{0.4}$$

and then we arrive at the sideways motion constraint by projecting the velocity of the robot into the direction of b_y

$$b_y^T \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = 0 \implies \begin{bmatrix} -\sin\theta & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \tag{0.5}$$

(b) Setting our system inputs as $u_1 = v$ and $u_2 = \omega$, express the dynamics of the system in the form

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

Where $g_1(q)$ and $g_2(q)$ form a basis for the null space of $A(q)$
A vector in the direction of b_x should give us something in the null space, and we also see that any non-zero $\dot{\theta}$ is also allowed. This yields

$$\dot{q} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2 \tag{0.6}$$

enumerate

Problem 2 - Lie Brackets

The *Lie Bracket*, or commutator, describes the degree to which two elements *commute* under some operation. For vector fields, the Lie Bracket is defined as

$$[f(q), g(q)] = \frac{\partial g(q)}{\partial q} f(q) - \frac{\partial f(q)}{\partial q} g(q)$$

The Lie Bracket has the following properties:

- Anti-symmetry (skew-symmetry): $[X, Y] = -[Y, X]$
- The Jacobi Identity: $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$.

i. Express the Lie Bracket using Lie derivatives (this is a bit of an abuse of notation, but it's fine for our purposes).
We have

$$\begin{aligned} [f(q), g(q)] &= \frac{\partial g(q)}{\partial q} f(q) - \frac{\partial f(q)}{\partial q} g(q) && (0.7) \\ &= L_f g - L_g f && (0.8) \end{aligned}$$

ii. Prove that the Lie Bracket is anti-symmetric.
We have

$$\begin{aligned} [f(q), g(q)] &= \frac{\partial g(q)}{\partial q} f(q) - \frac{\partial f(q)}{\partial q} g(q) && (0.9) \\ &= -(\frac{\partial f(q)}{\partial q} g(q) - \frac{\partial g(q)}{\partial q} f(q)) && (0.10) \\ &= -[g(q), f(q)] && (0.11) \end{aligned}$$

iii. (At home) Prove that the Lie Bracket satisfies the Jacobi Identity.
We have

$$\begin{aligned} & [f(q), [g(q), h(q)]] + [h(q), [f(q), g(q)]] + [g(q), [h(q), f(q)]] && (0.12) \\ &= \frac{\partial [g(q), h(q)]}{\partial q} f(q) - \frac{\partial f(q)}{\partial q} [g(q), h(q)] + \frac{\partial [f(q), g(q)]}{\partial q} h(q) - \frac{\partial h(q)}{\partial q} [f(q), g(q)] && (0.13) \\ &+ \frac{\partial [h(q), f(q)]}{\partial q} g(q) - \frac{\partial g(q)}{\partial q} [h(q), f(q)] && (0.14) \\ &= L_f L_g h - L_f L_h g - (L_g L_h f - L_h L_g f) + L_h L_f g - L_h L_g f - (L_f L_g h - L_g L_f h) && (0.15) \\ &+ L_g L_h f - L_g L_f h - (L_h L_f g - L_f L_h g) && (0.16) \\ &= (L_f L_g h - L_f L_g h) + (L_f L_h g - L_f L_h g) + (L_g L_h f - L_g L_h f) + (L_h L_g f - L_h L_g f) && (0.17) \\ &+ (L_h L_f g - L_h L_f g) + (L_g L_f h - L_g L_f h) && (0.18) \\ &= 0 && (0.19) \end{aligned}$$

iv. Lets define $q = (x, y, z)$

$$g_1(q) = \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix}, \quad g_2(q) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

A. What is $[g_1, g_2]$? Is it linearly independent from g_1, g_2?

$$\begin{aligned} [g_1, g_2] = &= \frac{\partial g_2(q)}{\partial q} g_1(q) - \frac{\partial g_1(q)}{\partial q} g_2(q) && (0.20) \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} && (0.21) \\ &= - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} && (0.22) \end{aligned}$$

We see that this is linearly independent from g_1, g_2.

B. What is $[g_1, [g_1, g_2]]$? Is it linearly dependent from $g_1, g_2, [g_1, g_2]$.?

$$[g_1, [g_1, g_2]] = \frac{\partial [g_1, g_2]}{\partial q} g_1(q) - \frac{\partial g_1(q)}{\partial q} [g_1, g_2] \tag{0.23}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{0.24}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{0.25}$$

We see that this is not linearly independent from $g_1, g_2, [g_1, g_2]$ enumerate

C. Lets define $q = (x, y, z)$

$$g_1(q) = \begin{bmatrix} xz \\ yz \\ 0 \end{bmatrix}, \quad g_2(q) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

A. What is $[g_1, g_2]$? Is it linearly independent from g_1, g_2?

$$[g_1, g_2] = \frac{\partial g_2(q)}{\partial q} g_1(q) - \frac{\partial g_1(q)}{\partial q} g_2(q) \tag{0.26}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} xz \\ yz \\ 0 \end{bmatrix} - \begin{bmatrix} z & 0 & x \\ 0 & z & y \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{0.27}$$

$$= - \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \tag{0.28}$$

e see that this is not linearly independent from g_1, g_2.

B. What is $[g_1, [g_1, g_2]]$? Is it linearly independent from $g_1, g_2, [g_1, g_2]$?

$$[g_1, [g_1, g_2]] = \frac{\partial [g_1(q), g_2(q)]}{\partial q} g_1(q) - \frac{\partial g_1(q)}{\partial q} [g_1(q), g_2(q)] \tag{0.29}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} xz \\ xz \\ 0 \end{bmatrix} + \begin{bmatrix} z & 0 & x \\ 0 & z & y \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \tag{0.30}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{0.31}$$

We see that this is not linearly independent from $g_1, g_2, [g_1, g_2]$. enumerate

Problem 3 - Constraint Identification

Given a physical system, you should be able to predict whether the constraints are holonomic or nonholonomic. Consider the following systems. What are the constraints? Are they holonomic or not? How many parameters do you need to minimally represent the system?

A.

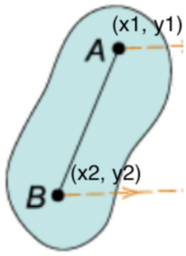

Figure 3: A planar rigid body

- Constraint: Distance between A, B is constant. Given point A, we can determine point B by the angle of vector $\vec{AB}$.
- Holonomy: This is a holonomic system. We can represent this with 3 parameters - x, y, θ. This is equivalent to an $SE(2)$ parameterization of a planar rigid body.

B.

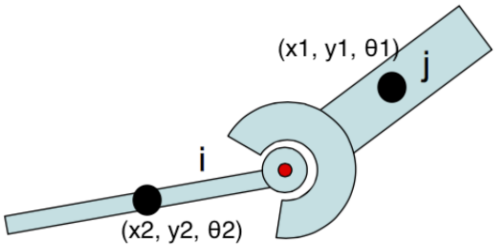

Figure 4: Two planar rigid bodies connected by a revolute joint

- Constraint: i is constrained to lie on a circle whose center is a fixed distance from j. Given pose j, we can determine pose i given we have θ_1 and θ_2 through forward kinematics.
- Holonomy: This is a holonomic system. We can represent this with 4 parameters - x, y, θ_1, θ_2, since all we need is pose and joint angle to recover both planar rigid bodies.

C.

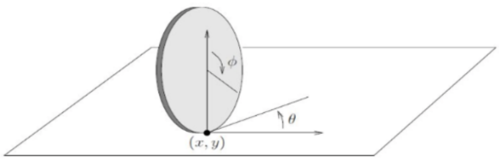

Figure 5: A disk rolling on a plane

- Constraint: No sideways motion (orthogonal to $\dot{x}, \dot{y}$ and no rolling without slipping ($\dot{x}, \dot{y}$ need to correspond with how fast we are rotating $\dot{\phi}$ and heading θ.
- Holonomy: The system is non-holonomic. We can reach any section of the state space.

D.

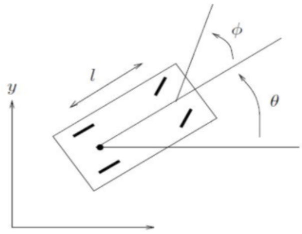

Figure 6: A bicycle model

- Constraint: No motion orthogonal to the front wheels, and no motion orthogonal to the back wheels.
- Holonomy: The system is non-holonomic. We can reach any section of the state space.

E.

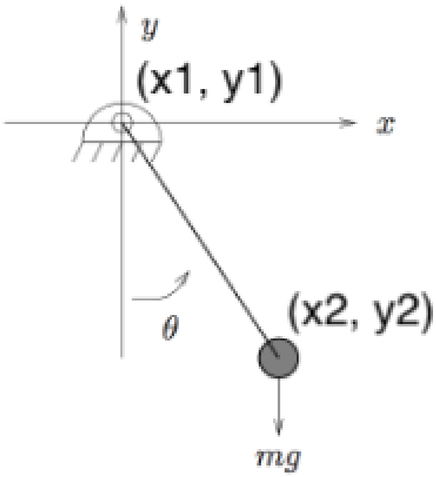

Figure 7: A simple pendulum

- Constraint: The pendulum (x_2, y_2) must always remain a fixed distance r from the origin (x_1, x_2).
- Holonomy: This is a holonomic system. We can represent the system with 3 parameters - x_1, x_2, θ, since the position of the pendulum can be recovered from these.

F.

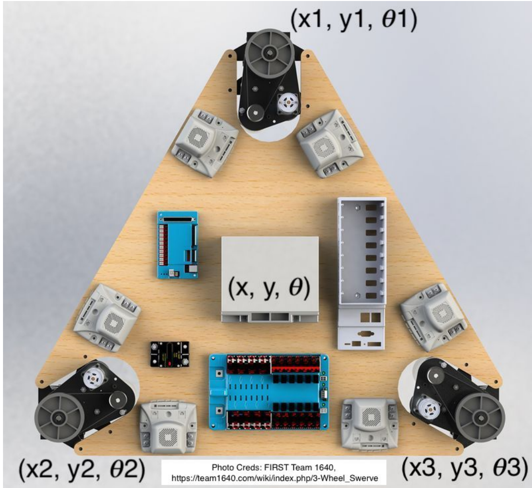

Figure 8: A swerve drive with three wheels

- Constraint: The distance between all of the wheels and the center of the robot is constrained - given the robot's x, y, θ, we can recover the positions of the wheels (but not their orientations).
- Holonomy: This is a holonomic system. We can represent the system with 6 parameters - $x, x, \theta, \theta_1, \theta_2, \theta_3$, since the position of the pendulum can be recovered from these.

G.

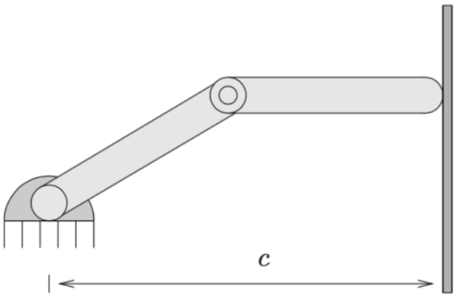

Figure 9: A robot touching a wall

- Constraint: It seems like many of the system's parameters have been obscured, so we are left with only the distance to the wall c as our parameter..
- Holonomy: This system is non-holonomic since we have no apparent restrictions on c (given our arm is long enough).