C106B Discussion 3 Walkthrough

Stability
Control
Nonlinear Systems

$\dot{x} = f(x)$

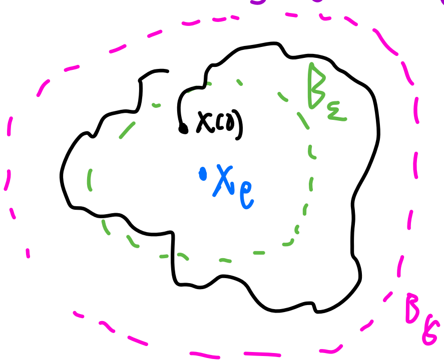
$= f(x, u)$

$x(t)$ goes to x_e as $t \to \infty$ Stability

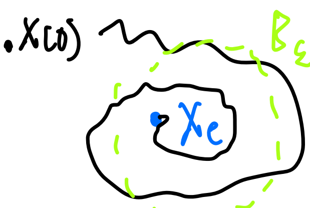
$x(t)$ tracks $x_d(t)$ Controllability

$x_e = 0$, $\tilde{y}(t) = x(t) - x_d(t)$

$\tilde{y}(t) \to x_e \Rightarrow x(t) = x_d(t)$



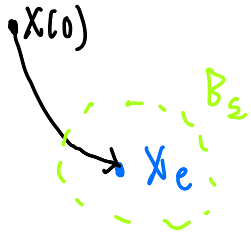
Stable in the Sense of Lyapunov



Asymptotic Stability

Any $x(0) \Rightarrow$ Global Asymptotic Stability

Within $B_\varepsilon \Rightarrow$ Local Asymptotic Stability



Exponential Stability

$$|x(t) - x_e| \le e^{\alpha t}|x(0) - x_e|$$

$\dot{x} = Ax$,

Linear System

$\max(Re(\sigma(A))) < 0$
$\Rightarrow$ Exponential Stability

$\dot{x} = \hat{\omega} x$ $\qquad$ $Re(\sigma(\hat{\omega})) = 0$

axis of rotation

Stability in the sense of Lyapunov

C106B - Robotic Manipulation and Interaction (Week 3)

Discussion #3

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Problem 1 - Lyapunov's Indirect Method: Modified Van Der Pol Oscillator

Consider the following model for an oscillator with nonlinear damping.

$$\ddot{x} + \mu(1 - x^2)\dot{x} + x = 0 \tag{0.1}$$

where μ is a scalar damping coefficient.

1. By choosing a good set of state variables, write the above model in state space form.

$X = X_1$

$\dot{X} = X_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - \mu(1 - x_1^2)x_2 \end{bmatrix}$$

2. Find all equilibria of this system.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - \mu(1 - x_1^2)x_2 \end{bmatrix} \qquad x_2 = 0 \qquad -x_1 = 0 \Rightarrow x_1 = 0$$

$X_e = (0, 0)$

3. Linearize the system about the equilibria. Using the indirect method of Lyapunov, comment on the stability of the equilibria for the cases where $\mu > 0$ and $\mu = 0$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \approx \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}\Bigg|_{x_e} + J\Big|_{x_e}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{x_1} & \frac{\partial f_2}{x_2} \end{bmatrix} \qquad J = \begin{bmatrix} 0 & 1 \\ -1 + 2\mu x_1 x_2 & -\mu(1 - x_1^2) \end{bmatrix}$$

$$J\Big|_{x_e} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu \end{bmatrix} \qquad \lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4}}{2}$$

$\mu > 0$: Local Asymptotic Stability

$\lambda|_{\mu=0} = \pm j$

$\mu = 0$: Stability in sense of Lyapunov

Problem 2 - Lyapunov's Direct Method: Unicycle Model Robot

Consider the following model for a unicycle model robot. The state is (x, y, θ) which represents the position of the center of the robot relative to some fixed origin along with its current heading. The control inputs are the linear velocity v and the angular velocity ω.

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}v\cos\theta\\ v\sin\theta\\ \omega\end{bmatrix} \tag{0.2}$$

In this problem, we will explore a technique called *point-offset* control for controlling unicycle model robots like the Turtlebot. Consider a point p attached rigidly to the robot at a distance δ from the center, in front of the robot (see figure 0.7). So, the position of p is given by:

$$\begin{bmatrix}p_x\\ p_y\end{bmatrix} = \begin{bmatrix}x + \delta\cos\theta\\ y + \delta\sin\theta\end{bmatrix} \tag{0.3}$$

Now consider the problem of driving the turtlebot to some neighbourhood of the origin. Instead of driving the turtlebot directly, we will instead attempt to control the robot so that the point p goes to the origin. Then, the turtlebot will be in a neighbourhood of radius δ around the origin. In the next few problems, we will develop a control law to drive p to the origin, and prove its stability.

1. Let the body frame axes of the turtlebot be $b_x = (\cos\theta, \sin\theta)^T$ and $b_y = (-\sin\theta, \cos\theta)^T$, as shown in figure 0.7. Show that

$$\dot{p} = vb_x + \delta\omega b_y \tag{0.4}$$

$$p = \begin{bmatrix}x + \delta\cos\theta\\ y + \delta\sin\theta\end{bmatrix}, \quad \dot{p} = \begin{bmatrix}\dot{x} + \delta -\sin\theta\,\dot{\theta}\\ \dot{y} + \delta\cos\theta\,\dot{\theta}\end{bmatrix}$$

$$= v\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} + \delta w\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$$\dot{p} = v b_x + \delta w b_y$$

2. Say we apply the following feedback control law on the robot:

$$v = -b_x^T p, \quad \omega = -\frac{1}{\delta}b_y^T p \tag{0.5}$$

Using the Lyapunov function

$$V = \frac{1}{2}p^T p = \frac{1}{2}\|p\|^2 \tag{0.6}$$

show that the point p converges asymptotically to the origin. Is the stability global?

$\dot{V} < 0$ (except when $\dot{V}(0) = 0$)

$$V = \frac{1}{2}p^T p, \quad \dot{V} = p^T\dot{p}$$

$\dot{V}(p) < 0$ (unless $p = 0$)

$$= p^T(v b_x + w\delta b_y)$$

Global Asymptotic Stability

$$= p^T((-b_x^T p)b_x + \delta(-\tfrac{1}{\delta}\cdot b_y^T p)b_y)$$

$$= -p^T[(b_x^T p)b_x + (b_y^T p)b_y]$$

$$= -p^T\begin{bmatrix}b_x & b_y\end{bmatrix}\begin{bmatrix}b_x^T\\ b_y^T\end{bmatrix}p = -p^T p$$

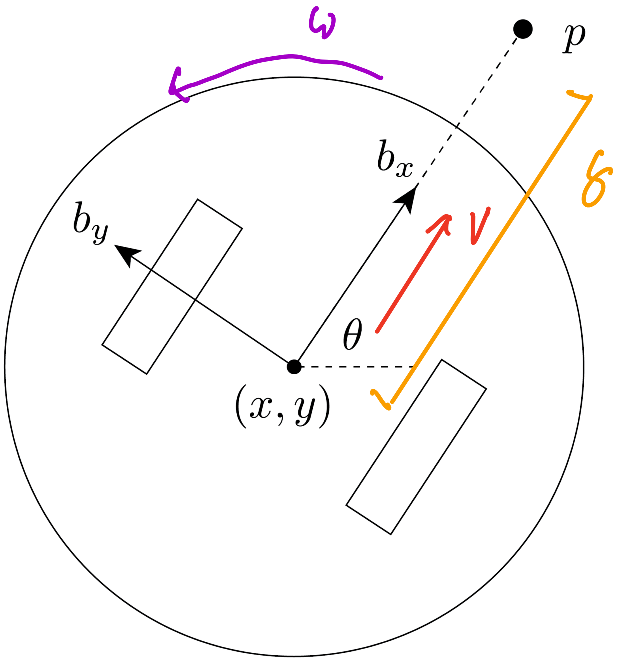
$$= -\|p\|^2$$

3. Is it exponentially stable? If so, is the stability global?

$\dot{V} = -\|p\|^2$, $\dot{V} = -2V$

$V = \frac{1}{2}\|p\|^2$ $\quad V(t) = e^{-2t} V(0)$

Global Exponential Stability



(0.7)

$$\dot{x}(t) = f(x(t))$$

As $t \to \infty$, $x(t) \to x_e$

Want to show asymptotic stability

$V(x)$:

"energy function"

- $V(x) \geq 0$, $V(x) = 0 \iff x = x_e$
- $V: \mathbb{R}^n \to \mathbb{R}$
- $\dot{V}(x) \leq 0$, $\dot{V}(x) = 0 \iff x = x_e$