C106B - Robotic Manipulation and Interaction **(Week 2)**

Discussion #2

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Problem 1 - Velocities are Twists, Twists are Velocities

Consider a rigid body (body frame B, spatial frame A) performing a uniform twist motion $\xi = (v, \omega)$, so that its configuration at time t is given by:

$$g_{ab}(t) = e^{\hat{\xi}t} g_{ab}(0) \tag{0.1}$$

1. Let p_a be the spatial coordinates for a point on the rigid body. While the body is performing this motion, write down an expression for $\dot{p}$ in terms of p and the coordinates of the twist ξ.
   There are multiple ways to approach this question, but one is to use our motivation of a twist ξ as defining the differential equation

   $$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \widehat{\xi} \begin{bmatrix} p \\ 1 \end{bmatrix} \tag{0.2}$$

   For a twist $\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}$, we get

   $$\dot{p} = \widehat{\omega} p + v \tag{0.3}$$

2. Compute the spatial rigid body velocity $\hat{V}_{ab}^s$ of the frame $g_{ab}(t)$ as a function of time (Recall the derivative of a matrix exponential: $d(e^{At})/dt = Ae^{At}$).
   We employ the definition of the spatial rigid body velocity to get

   $$\begin{aligned} \widehat{V_{ab}^S}(t) &= \dot{g}(t) g^{-1}(t) && (0.4)\\ &= \widehat{\xi} e^{\widehat{\xi}t} g_{ab}(0) g_{ab}^{-1}(0) e^{-\widehat{\xi}t} && (0.5)\\ &= \widehat{\xi} && (0.6)\\ & && (0.7) \end{aligned}$$

   Intuitively, this should make sense this our trajectory is generated by the evolution of a constant screw motion.

3. Compute the body velocity $\hat{V}_{ab}^b$ of the frame $g_{ab}(t)$ as a function of time.
   We employ the definition of the body rigid body velocity to get

   $$\begin{aligned} \widehat{V_{ab}^B}(t) &= g^{-1}(t) \dot{g}(t) && (0.8)\\ &= g_{ab}(0) g_{ab}^{-1}(0) e^{-\widehat{\xi}t} \widehat{\xi} e^{\widehat{\xi}t} && (0.9)\\ &= g_{ab}(0) g_{ab}^{-1}(0) e^{-\widehat{\xi}t} e^{\widehat{\xi}t} \widehat{\xi} && (0.10)\\ &= \widehat{\xi} && (0.11)\\ & && (0.12) \end{aligned}$$

   Intuitively, this should make sense this our trajectory is generated by the evolution of a constant screw motion.

4. Interpret screw motions as simply the analog of moving with a constant velocity, and hence interpret twists as velocities.

   For a $SE(3)$ trajectory given by constrant screw motion $\xi \in se(3)$, the rigid body velocity associated with that trajectory at every point in time $V(t)$ is given by nothing but ξ.

Problem 2 - Workspace Tracking with Jacobians

We would like the end effector of our robot arm to perform some trajectory in its workspace. This trajectory is given to us as a trajectory of rigid transforms, $g(t)$. Assume we have access to both $g(t)$ and $\dot{g}(t)$ for $t \in [0, T]$, and that the trajectory starts from the current position of the robot, $g(0)$. Recall that we can only give the robot jointspace commands, so we want to convert this workspace trajectory into a jointspace trajectory $\theta(t)$ such that $g_{ST}(\theta(t)) = g(t)$ for $t \in [0, T]$ where $g_{ST}(\theta)$ is the forward kinematics map.

1. How would you solve this problem using an inverse kinematics solver? Why might this be undesirable?
   Assume we have some magic IK function
   $$\text{IK} : SE(3) \to \mathbb{R}^n \tag{0.13}$$
   We can discretize our time interval into points evenly spaced Δt apart, so $g_i = g(t \cdot \Delta t)$. For each g, we can compute the associated vector of joint angles to get
   $$\theta_i = \text{IK}(g_i) \tag{0.14}$$
   However, this approach isn't perfect. Some concerns include
   - Computation: Is it efficient to compute IK at every timestep?
   - Continuity: Will this trajectory be continous? Smooth?
   - Feasibility: What happens when our IK solver doesn't find a solution
2. Write down an expression for the desired spatial workspace velocity $\hat{V}$ that we want the end effector to perform at time t, in order to perform the trajectory g. Using the definition of rigid body velocity, we have
   $$V^s = \dot{g}g^{-1} \tag{0.15}$$
3. Assume we also have efficient access to the spatial jacobian J. Write down an expression for $\dot{\theta}(t)$ in terms of the velocity you computed in the previous part (recall the Moore-Penrose pseudoinverse that we spoke about in lecture). Recall the relationship between the manipulator Jacobian and end effector rigid body velocity
   $$J\dot{\theta} = V \tag{0.16}$$
   We can find a (minimum-norm) solution to this equation by computing
   $$\dot{\theta}(t) = J_s^\dagger(\theta(t))V^s(t) \tag{0.17}$$
   where $J_s^\dagger$ refers to the Moore-Penrose psuedoinverse of the spatial jacobian.
4. Write down an expression for $\theta(t)$, using your answers to the previous two parts. Calculus helps us out here
   $$\theta(t) = \int_0^t J_s^\dagger(\theta(t))V^s(t)\text{d}t + \theta(0) \tag{0.18}$$
   Note that this gives us a smooth, continuous trajectory for $\theta(t)$. But as you will see in Project 1, this isn't a perfect approach!
5. Can we use the previous part to do inverse kinematics? Yes! We can just perform the above procedure after defining our $g(t)$ trajectory as follows
   - $g(0)$ will be the current end effector pose of the manipulator
   - $g(T)$ will be the desired end effector pose of the manipulator
   - Define $g(t) = e^{\hat{\xi}t}g(0)$, where $\hat{\xi} = \frac{\log g(T)g^{-1}(0)}{t}$

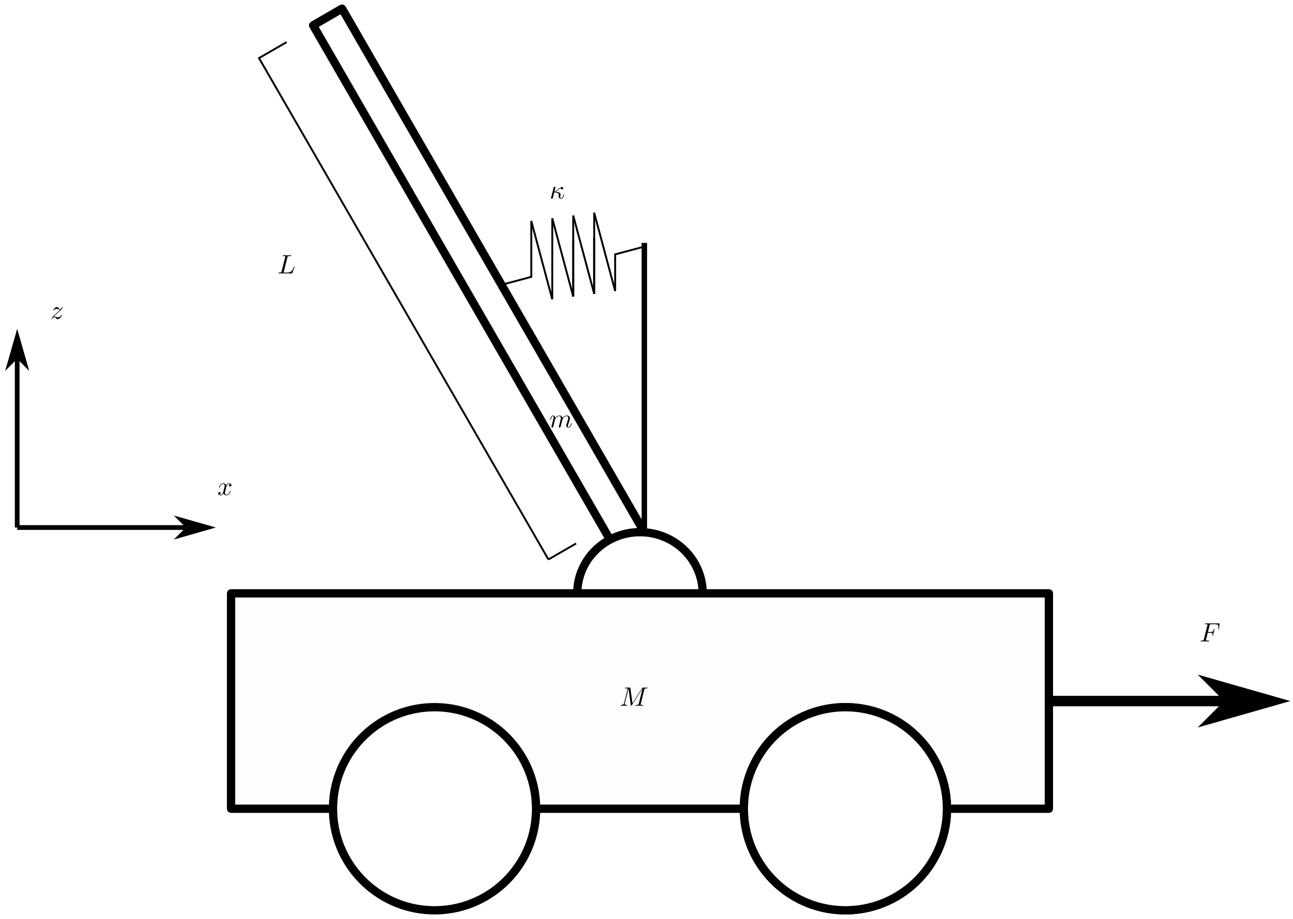

Figure 1: Cart-Pendulum System

Problem 3 - Lagrangian Dynamics

A cart with mass M is free to move in the x-direction without resistance. A uniform rod of length L with mass m and moment of inertia I (about its center of mass) is mounted on a frictionless pivot on top of the cart. An external force F is applied as an input. The pivot also has mounted in it a torsional spring with spring constant κ, as shown. Pick a suitable set of generalized coordinates, and find the equations of motion of the system in terms of those coordinates using Lagrangian dynamics.

First, we define our generalized coordinates as

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix} \tag{0.19}$$

since we can uniquely describe the state of our system using the position of the car and the angle of rotation of the beam. In order to calculate the kinetic energy of the system, we will need the velocity of the center of the mass of the rod. This will be

$$\dot{v} = \begin{bmatrix} \dot{x} - \dot{\theta}\cos\theta \\ -\dot{\theta}\sin\theta \end{bmatrix} \tag{0.20}$$

The magnitude of this velocity is then

$$||\dot{v}||^2 = \dot{x}^2 + \dot{\theta}^2 - 2\dot{x}\dot{\theta}\cos\theta \tag{0.21}$$

The other kinetic energy terms are more straight forward to derive. This puts the total kinetic energy of the system as

$$T = \underbrace{\frac{1}{2}M\dot{x}^2}_{\text{kinetic energy of cart}} + \underbrace{\frac{1}{2}m(\dot{x}^2 + \dot{\theta}^2 - 2\dot{x}\dot{\theta}\cos\theta)}_{\text{translational kinetic energy of rod}} + \underbrace{\frac{1}{2}I\dot{\theta}^2}_{\text{rotational kinetic energy of rod}} \tag{0.22}$$

Now, we need to consider the potential energy of the system.

$$V = \frac{1}{2}\kappa\theta^2 + mg(L/2 \cdot \cos\theta) \tag{0.23}$$

The Lagrangian of the system is given by

$$L = T - V \tag{0.24}$$

which gives us equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \begin{bmatrix} F \\ 0 \end{bmatrix} \tag{0.25}$$