EECS C106B - Robotic Manipulation and Interaction **(Week 1)**

Discussion #1

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Problem 1 - Matrix Exponential and Linear ODEs

Recall that the exponential of a square matrix $A \in \mathbb{R}^{n\times n}$ is defined by the following infinite series:

$\dot{x} = Ax$, $x(t) = e^{At}x(0)$

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{0.1}$$

$A + (At)$ ...

1. Let $Y(t) = e^{At}$. By differentiating the series representation, show that $\dot{Y}(t) = Ae^{At}$.

   $e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$ $\quad A \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = Ae^{At}$

2. Show that $(e^A)^{-1} = e^{-A}$

   $e^A e^B = e^{A+B}$ when A, B commute

3. Show that $x(t) = e^{At}x_0$ is the *unique* solution to the differential equation $\dot{x} = Ax$ with initial condition $x(0) = x_0$, for $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$.
   Hint: Do this by first considering the function $y(t) = e^{-At}x(t)$. What is the time derivative of $y(t)$?

Problem 2 - Rigid Body Potpourri

1. Write the expressions for the velocity of the point p (ie. $\dot{p}(t)$) when attached to both the revolute and prismatic joints in Fig. 1. Assume that $\omega \in \mathbb{R}^3$, $||\omega|| = 1$, and $q \in \mathbb{R}^3$ is some point along the axis of ω.

2. Find $\widehat{\xi}$ to complete the following expression of $\dot{p}(t)$ in homogeneous coordinates for a revolute joint.

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \quad\quad\quad\quad\quad\quad \end{bmatrix}}_{=:\,\widehat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

   Hint: Recall the skew symmetric matrix $\widehat{w}$ of w:

$$\omega = [\omega_1 \;\; \omega_2 \;\; \omega_3]^T; \quad \widehat{w} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{0.2}$$

3. Find $\widehat{\xi}$ to complete the following expression of $\dot{p}(t)$ in homogeneous coordinates for a prismatic joint.

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \quad\quad\quad\quad\quad\quad \end{bmatrix}}_{=:\,\widehat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

4. Write the general solution to the differential equation $\dot{\bar{p}} = \widehat{\xi}\bar{p}$. Then, make use of the fact that $||\omega|| = 1$ to reparameterize t to be θ. Specifically, find the expression for $p(\theta)$ in terms of $p(0)$.

5. Recall that a screw motion S consists of an axis l, a pitch h, and a magnitude M. The transformation g corresponding to S has the following effect on a point p

$$gp = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega \tag{0.3}$$

   Convert this transformation to homogeneous coordinates.

$$\underbrace{\frac{A^n}{n!}}_{\text{constant}} t^n = \frac{A^n}{(n-1)!} t^{n-1} = A \underbrace{\frac{A^{n-1}}{(n-1)!}}_{(n-1)^{th} \text{ term of the sum}} t^{n-1}$$

$$\frac{n}{n!} = \frac{1}{(n-1)!}$$

$$A \cdot -A = -A \cdot A$$

$$e^{A} e^{-A} = e^{A-A} = e^{0} = I$$

$$\dot{y}(t) = -Ae^{-At}x(t) + e^{-At} \cdot (Ax(t))$$

$$= -Ae^{-At}x(t) + Ae^{-At}x(t)$$

$$= 0$$

$$y(t) = e^{-At}x(t) = x_0, \quad x(t) = e^{At}x_0$$

$$y(0) = x(0) = x_0$$

$\dot{x} = Ax$

$x(t) = e^{At} x_0$

$A = \hat{w}$, $\quad \hat{w} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$

Axis of rotation

$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

$\hat{w} \in so(3)$

$e^{\hat{w} t} = R \in \underbrace{SO(3)}_{\text{special orthogonal}}$

$\xi \in \mathbb{R}^6$, $\quad \xi \in se(3)$ $\quad \xi = \begin{bmatrix} v \\ w \end{bmatrix}$

$\hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix}$

$e^{\hat{\xi} t} = g \in SE(3)$

Log:

e: $\left[\begin{array}{l} se(3) \to SE(3) \\ so(3) \to SO(3) \end{array} \right]$ Lie Groups

Lie Algebra

Problem 3 - Forward Kinematics

Fig. 2 shows the initial configuration of a robot arm whose first joint is a screw joint of pitch h. The other two joints are revolute

1. Write down the 4×4 initial end effector configuration of the manipulator $g_{sb}(0)$
2. Find the twists ξ_1, ξ_2, ξ_3 corresponding to each of the joints of the manipulator, and hence write down an expression for the forward kinematics map $g_{sb}(\theta)$. You may leave your answer in terms of the exponentials of known matrices.

Problem 4 - Inverse Kinematics

Consider a manipulator where we have the forward kinematics map

$$g_{st}(\theta) : \theta \to SE(3) \tag{0.4}$$

1. Use the g_{st} map to explain what it means for the manipulator to have multiple IK solutions for some end effector configuration.
2. Why can multiple IK solutions be good? Bad?

g_T θ_1, θ_2

$g_{st}(\theta_1) = g_{st}(\theta_2) = g_T$

$\theta_1 \neq \theta_2$

$\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$

$\Theta \xrightarrow{g_{st}} \begin{bmatrix} x \\ y \\ T \end{bmatrix}$

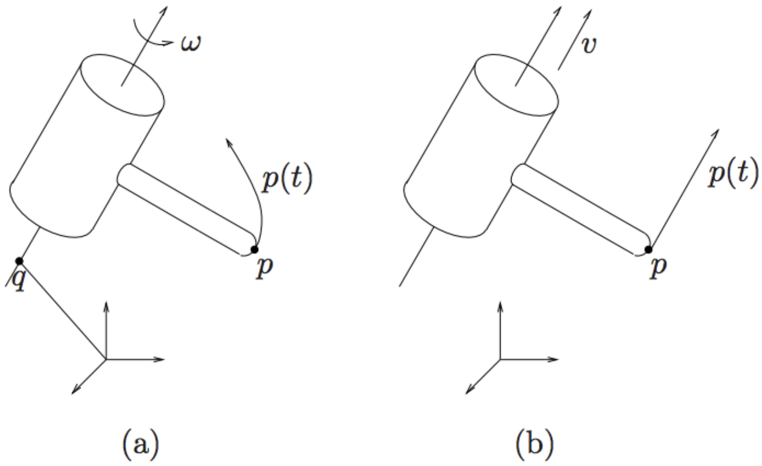

Figure 1: a) A revolute joint and b) a prismatic joint.

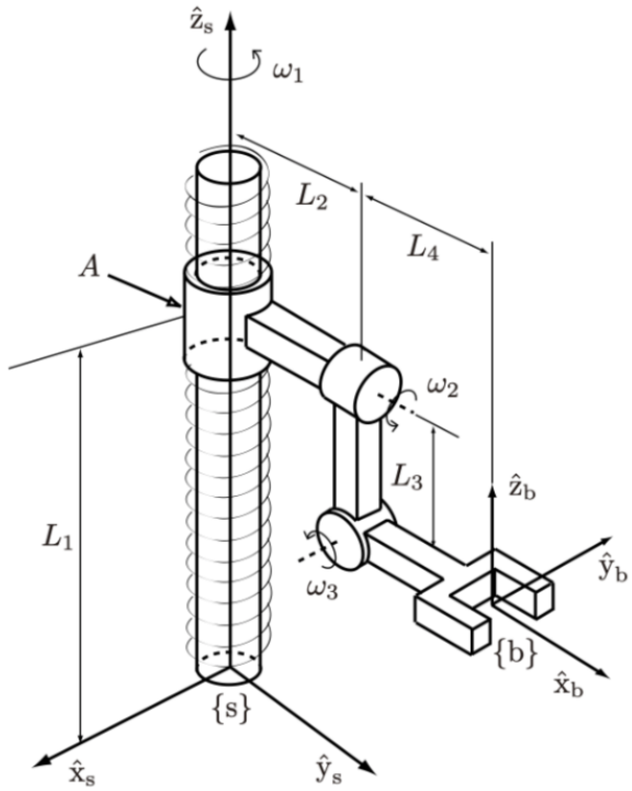

Figure 2: A 3DOF manipulator