EECS C106B - Robotic Manipulation and Interaction **(Week 1)**

Discussion #1

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Problem 1 - Matrix Exponential and Linear ODEs

Recall that the exponential of a square matrix $A \in \mathbb{R}^{n\times n}$ is defined by the following infinite series:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{0.1}$$

1. Let $Y(t) = e^{At}$. By differentiating the series representation, show that $\dot{Y}(t) = Ae^{At}$.
   We can write
   $$Y(t) = e^{At} = \sum_{n=0}^{\infty} \frac{A^n}{n!} t^n \tag{0.2}$$
   Now differentiate with respect to t to get
   $$\dot{Y}(t) = \sum_{n=0}^{\infty} \frac{A^n}{n!} n t^{n-1} = \sum_{n=1}^{\infty} \frac{A^n}{(n-1)!} t^{n-1} \tag{0.3}$$
   Now from the above series we can factor out an A in two ways:
   $$\dot{Y}(t) = A\left(\sum_{n=1}^{\infty} \frac{A^{n-1}}{(n-1)!} t^{n-1}\right) = Ae^{At} \tag{0.4}$$
   $$\dot{Y}(t) = \left(\sum_{n=1}^{\infty} \frac{A^{n-1}}{(n-1)!} t^{n-1}\right) A = e^{At}A \tag{0.5}$$
   $$\tag{0.6}$$
   as needed.

2. Show that $(e^A)^{-1} = e^{-A}$
   Consider two square matrices A and B. If two square matrices commute, i.e. $AB = BA$, then we can write
   $$e^A e^B = e^{A+B} \tag{0.7}$$
   Because A and $-A$ always commute,
   $$e^A e^{-A} = e^{A+(-A)} = e^0 = I \tag{0.8}$$
   Therefore
   $$(e^A)^{-1} = e^{-A} \tag{0.9}$$

3. Show that $x(t) = e^{At}x_0$ is the *unique* solution to the differential equation $\dot{x} = Ax$ with initial condition $x(0) = x_0$, for $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$.
   Hint: Do this by first considering the function $y(t) = e^{-At}x(t)$. What is the time derivative of $y(t)$?
   Consider the time derivative $\dot{y}$. Using the chain rule, we can write
   $$\begin{aligned}\dot{y}(t) &= e^{-At}\dot{x}(t) - Ae^{-At}x(t) \\ &= e^{-At}Ax(t) - Ae^{-At}x(t) \\ &= (e^{-At}A - Ae^{-At})x(t) \\ &= 0\end{aligned}$$

since $\dot{y}$ is uniformly 0 and y is clearly differentiable, y must be some constant vector $c \in \mathbb{R}^n$. So we have $e^{-At}x(t) = c \to x(t) = e^{At}c$. By substituting $t = 0$, it is clear that $c = x(0) = x_0$, giving us the unique solution for $x(t) = e^{At}x_0$ as needed.

Problem 2 - Rigid Body Potpourri

1. Write the expressions for the velocity of the point p (ie. $\dot{p}(t)$) when attached to both the revolute and prismatic joints in Fig. 1. Assume that $\omega \in \mathbb{R}^3$, $||\omega|| = 1$, and $q \in \mathbb{R}^3$ is some point along the axis of ω.

   For the revolute joint: Recall that ωr produces the linear velocity of a point rotating about axis ω with radius r. But in the null space of $\widehat{\omega}$ is any vector along the axis of ω. So we may replace r with any vector that can be written as $r + c \cdot \omega$. This leads us to the expression $\dot{p}(t) = \omega \times (p(t) - q)$.

   For the prismatic joint: Not having any rotation simplifies things severely! We just consider our linear component and have $\dot{p}(t) = v$.

2. Find $\widehat{\xi}$ to complete the following expression of $\dot{p}(t)$ in homogeneous coordinates for a revolute joint.

   We can expand out our expression for linear velocity to get

$$\dot{p} = \widehat{\omega}p - \widehat{\omega}q \tag{0.10}$$

   Factor terms into a matrix-vector expression to get the desired result.

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \widehat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix}}_{=: \widehat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

   Hint: Recall the skew symmetric matrix $\widehat{w}$ of w:

$$\omega = [\omega_1 \ \ \omega_2 \ \ \omega_3]^T; \ \ \widehat{w} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{0.11}$$

3. Find $\widehat{\xi}$ to complete the following expression of $\dot{p}(t)$ in homogeneous coordinates for a prismatic joint.

   We see that hte velocity does not depend on $\dot{p}$, so the only part of the $\widehat{\xi}$ matrix which becomes non-zero is the additive component.

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}}_{=: \widehat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

4. Write the general solution to the differential equation $\dot{\bar{p}} = \widehat{\xi}\bar{p}$. Then, make use of the fact that $||\omega|| = 1$ to reparameterize t to be θ. Specifically, find the expression for $p(\theta)$ in terms of $p(0)$.

   We make use of hte matrix exponential to solve our linear matrix differential equation. $||\omega|| = 1$ means that we have a unit angular velocity, so we can say that radians = seconds. This allows us to make a direct substitution for $\theta = t$.

$$\bar{p}(t) = e^{\widehat{\xi}t}\bar{p}(0)$$
$$\bar{p}(\theta) = e^{\widehat{\xi}\theta}\bar{p}(0)$$

   Recall that a screw motion S consists of an axis l, a pitch h, and a magnitude M. The transformation g corresponding to S has the following effect on a point p

$$gp = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega \tag{0.12}$$

   Convert this transformation to homogeneous coordinates.

$$g \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\widehat{\omega}\theta} & (I - e^{\widehat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Problem 3 - Forward Kinematics

Fig. 2 shows the initial configuration of a robot arm whose first joint is a screw joint of pitch h. The other two joints are revolute

1. Write down the 4×4 initial end effector configuration of the manipulator $g_{sb}(0)$

   By direct inspection we can write

$$g_{sb}(0) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & L_2 + L_4 \\ 0 & 0 & 1 & L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Find the twists ξ_1, ξ_2, ξ_3 corresponding to each of the joints of the manipulator, and hence write down an expression for the forward kinematics map $g_{sb}(\theta)$. You may leave your answer in terms of the exponentials of known matrices.

The first joint is a screw joint, and the others are standard revolute joints. So for the first joint we have $v_1 = -\omega_1 \times q_1 + h\omega_1$

$$q_1 = [0, 0, 0]^T, \omega_1 = [0, 0, 1]^T, v_1 = [0, 0, h]^T, \xi_1 = [0, 0, 2, 0, 0, 1]^T$$

For every other joint, we just have $v_i = -\omega_i \times q_i$

$$q_2 = [0, 0, L_1]^T, \omega_2 = [0, 1, 0]^T, v_2 = [-10, 0, 0]^T, \xi_2 = [-10, 0, 0, 0, 1, 0]^T$$
$$q_3 = [0, 5, 5]^T, \omega_3 = [1, 0, 0]^T, v_3 = [0, 5, -5]^T, \xi_3 = [0, 5, -5, 1, 0, 0]^T$$

Then the FK map is simply

$$g_{sb}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} g_{st}(0)$$

Problem 4 - Inverse Kinematics

Consider a manipulator where we have the forward kinematics map

$$g_{st}(\theta) : \theta \to SE(3) \tag{0.13}$$

1. Use the g_{st} map to explain what it means for the manipulator to have multiple IK solutions for some end effector configuration.

   If the manipulator has multiple IK solutions, that means for some end effector configuration $g \in SE(3)$, there exist two distinct vectors of joint angles $\theta_1, \theta_2, \theta_1 \neq \theta_2$ of appropriate dimension such that

$$g_s t(\theta_1) = g_s t(\theta_2) = g \tag{0.14}$$

2. Why can multiple IK solutions be good? Bad?

   Good: We have more flexibility when reaching goals for our manipulator (ie. We need our manipulator to hold a cup, but also need to ensure that the manipulator's elbow doesn't hit a table. Multiple IK solutions means that it's more likely that we have a joint angle configuration for our robot where the elbow is not in contact with the table).

   Bad: Path planning becomes harder when certain IK solutions for the same end effector configuration may put the manipulator in an awkward position for further manipulator even when there exists an IK solution with the desired manipulator configuration.

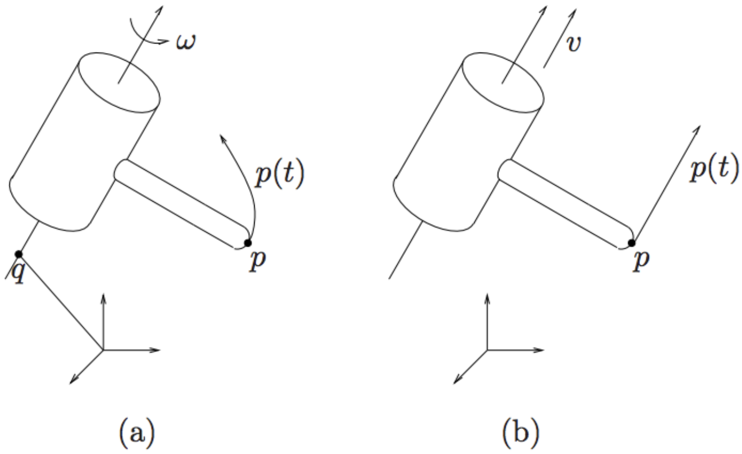

Figure 1: a) A revolute joint and b) a prismatic joint.

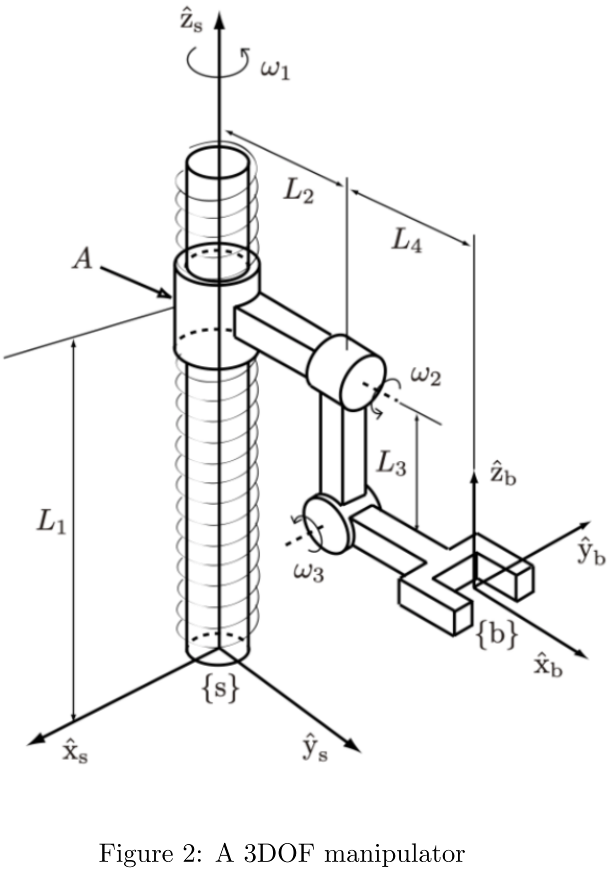

Figure 2: A 3DOF manipulator