Discussion 7 : Two-View Geometry

Author: Ritika Shrivastava, Haozhi Qi

7.1 Question 1: Fundamental Matrix

Let:

$$F = \begin{bmatrix} f_1 & f_4 & f_7 \\ f_2 & f_5 & f_8 \\ f_3 & f_6 & f_9 \end{bmatrix}$$

and let $\mathbf{f} = [f_1, ..., f_9]$. Show that the epipolar constraint can be represented as the following inner product:

$$a(x_i, x_i')\mathbf{f} = 0$$

where $a(x_i, x_i')$ is a row vector that depends only on the coordinates of the point x_i and x_i'

Note: $x_i' F x_i = 0$

Answer 1 *If we let* $\mathbf{x}_i = \begin{bmatrix} x_i & y_i & 1 \end{bmatrix}^T$ *and* $\mathbf{x}_i' = \begin{bmatrix} x_i' & y_i' & 1 \end{bmatrix}^T$*, by expanding* (1) *we obtain:*

$$(x_i'x_i)f_1 + (y_i'x_i)f_2 + x_if_3 + (x_i'y_i)f_4 + (y_i'y_i)f_5 + y_if_6 + x_i'f_7 + y_i'f_8 + f_9 = 0$$

that, in vector form can be written as:

$$\begin{bmatrix} x_i'x_i & y_i'x_i & x_i & x_i'y_i & y_i'y_i & y_i & x_i' & y_i' & 1 \end{bmatrix} \boldsymbol{f}$$

Hence we conclude that:

$$\boldsymbol{a}(\boldsymbol{x}_i, \boldsymbol{x}_i') = \begin{bmatrix} x_i'x_i & y_i'x_i & x_i & x_i'y_i & y_i'y_i & y_i & x_i' & y_i' & 1 \end{bmatrix}$$

7.2 Question 2: Fundamental Matrix Details

Answer the following questions, providing an adequate justification.

1. How many degrees of freedom does the fundamental matrix F have?
2. How many independent equations do we need to estimate f? Why?
3. Explain how the system of equations can be written in matrix form as Af = 0. What are the dimensions of A? Write explicitly A.
4. How would you estimate f via constrained least squares?

Answer 2 *Note how most of the considerations are very similar to those we made in order to estimate an homography.*

- *Since the constraint* (1) *holds in the projective space, F is defined up to a scale factor and therefore there are only 8 independent ratios. Moreover the rank of the matrix is 2 (i.e. the determinant is zero). Hence the fundamental matrix has 7 degrees of freedom.*

- *Since F has 7 degrees of freedom we need at least 7 points (plus two extra constraints that will fix the scale and enforce the rank condition). Note that the algorithm that we are developing is called 8 points algorithm and it uses at least 8 points, since the rank condition is not enforced explicitly.*
- *The set of equations for N point pairs can be written in matrix form as:*

$$A\boldsymbol{f} = \begin{bmatrix} \boldsymbol{a}(\boldsymbol{x}_1, \boldsymbol{x}'_1) \\ \boldsymbol{a}(\boldsymbol{x}_2, \boldsymbol{x}'_2) \\ \vdots \\ \boldsymbol{a}(\boldsymbol{x}_N, \boldsymbol{x}'_N) \end{bmatrix} \boldsymbol{f} = 0$$

  where $A \in \mathbb{R}^{N \times 9}$.

- *Similarly to what we did for homographies, we need to solve the constrained least square problem:*

$$\hat{\boldsymbol{f}} = \underset{\boldsymbol{f} \in \mathbb{R}^9, \|\boldsymbol{f}\|=1}{\operatorname{argmin}} \|A\boldsymbol{f}\|$$

7.3 Question 3: One more Time - Fundamental Matrix

Recall that the fundamental matrix can be written as $F = \hat{e'}K'RK^{-1}$. Show that $e' \times x' = Fx$.

Answer 4:

$$\begin{aligned} F\mathbf{x} &= [\mathbf{e}']_\times K'RK^{-1}\mathbf{x} \\ &= (\text{since } \mathbf{x} = P_1\mathbf{X}) \\ &= [\mathbf{e}']_\times K'RK^{-1}K \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \mathbf{X} \\ &= [\mathbf{e}']_\times K'R \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \mathbf{X} \\ &= [\mathbf{e}']_\times K' \begin{bmatrix} R & \mathbf{0} \end{bmatrix} \mathbf{X} \end{aligned}$$

We are almost there. Following the suggestion we recall that in Question 2 we showed that $\mathbf{e}' = K'\mathbf{t}$ and that the cross product of a vector with itself is zero. Therefore we have that:

$$\begin{aligned} F\mathbf{x} &= [\mathbf{e}']_\times K' \begin{bmatrix} R & \mathbf{0} \end{bmatrix} \mathbf{X} \\ &= [\mathbf{e}']_\times K' \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \mathbf{X} \\ &= [\mathbf{e}']_\times \mathbf{x}' \\ &= \mathbf{e}' \times \mathbf{x}' \end{aligned}$$

7.4 Question 4: Homographies

Let p correspond to a point on one image and let p correspond to the same point in the scene, but projected onto another image. Write a general equation for how a homography matrix H maps points from one image to another. How would H be restricted if it must describe an affine transformation?

The mapping is of the form $Hp = p'$. The affine transform is a subset of homographies where the third row of H is $[0\ 0\ 1]$. Your linear equation for a single point correspondence takes the following form:

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix}$$

Note that we are using homogeneous coordinates in this case, but our point correspondences are in image coordinates. In the affine homography case, $w = 1$.

7.5 Question 5: Homographies

How many points do you need to calculate a homography and why?

$$\text{Lets assume } x = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \text{ and } x' = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \text{ in homogeneous coordinates}$$

$$\text{and } H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}$$

$$c \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

By eliminating c we can formulate the above equation in the form

$$Ah = 0$$

$$\text{where } A = \begin{pmatrix} -x & -y & -1 & 0 & 0 & 0 & ux & uy & u \\ 0 & 0 & 0 & -x & -y & -1 & vx & vy & v \end{pmatrix}$$

$$\text{and } h = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 \end{pmatrix}^T$$

When there are more than 4 points it would be an over-determined case and hence we have to use least square solution to find h

$$min_h ||Ah||_2 \text{ rank(A)} = 3$$