EECS C106B / 206B Robotic Manipulation and Interaction **Spring 2022**

Discussion 7 : Two-View Geometry

Author: Ritika Shrivastava, Haozhi Qi

7.1 Question 1: Camera Projection Matrices

Consider the point X^{in} expressed with respect to the world coordinate system (or equivalently with respect to first camera coordinate system). Suppose that the transformation between the two cameras is described by the rotation matrix R and the translation vector t so that $X^{in\prime} = RX^{in} + t$.Assume that the first camera calibration matrix is K and that the second camera calibration matrix is K'. What are the camera projection matrices P and P' that allow us to write:

$$x = PX$$

$$x' = PX'$$

Answer 1: We have that:

$$\mathbf{x} = \underbrace{K \left[\begin{array}{cc} I & \mathbf{0} \end{array} \right]}_{P} \mathbf{X} = P\mathbf{X}$$

and:

$$\mathbf{x}' = \underbrace{K' \left[\begin{array}{cc} R & \mathbf{t} \end{array} \right]}_{P'} \mathbf{X} = P'\mathbf{X}$$

7.2 Question 2: Epipoles

Write the expression of the epipoles e and e'.

Answer 2: The epipole is the image of the optical center of one camera on the image plane of the other camera. Therefore (since $\mathbf{X}^{in} = R^{-1}\mathbf{X}^{in\ \prime} - R^{-1}\mathbf{t}$):

$$\mathbf{e} = P \begin{bmatrix} -R^{-1}\mathbf{t} \\ 1 \end{bmatrix} = -KR^{-1}\mathbf{t} \qquad \mathbf{e}' = P' \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = K'\mathbf{t}$$

7.3 Question 3: Recap

Consider two identical cameras (such that $K = K'$), whose optical axis coincides with the z axis, whose focal length is 0.03 m, with square pixels, no pinhole point offset and zero skew. What is the expression of K? Hint: K is the camera calibration matrix here.

$$K = \begin{bmatrix} 0.03 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7.4 Question 4: Understanding the Epipolar Line

What is the projective interpretation of $e' \times x'$? How does this object relate to the epipolar line l'?

Answer 3: The object $\mathbf{e}' \times \mathbf{x}'$ is a line in $\mathbb{P}^2$. More specifically it is the epipolar line $\mathbf{l}'$ associated to the point $\mathbf{x}$ on the image plane of the first camera, since it passes through the epipole $\mathbf{x}'$ and the projection of the point $\mathbf{X}$ on the image plane (of the second camera).

7.5 Question 5: Essential Matrix

Given the that first camera's 3D nodal point can be represented with d^L and the second camera's 3D nodal point can be represented with d^R. Let R^L be the rotation from the global frame of the first camera and R^R the same for the second camera. How can you represent the Essential Matrix for this setup?

$$E = R^L[\vec{d}_w^L - \vec{d}_w^R]_\times (R^R)^T.$$

7.6 Question 6: Fundamental Matrix

Let:

$$F = \begin{bmatrix} f_1 & f_4 & f_7 \\ f_2 & f_5 & f_8 \\ f_3 & f_6 & f_9 \end{bmatrix}$$

and let $\mathbf{f} = [f_1, ..., f_9]$. Show that the epipolar constraint can be represented as the following inner product:

$$a(x_i, x_i')\mathbf{f} = 0$$

where $a(x_i, x_i')$ is a row vector that depends only on the coordinates of the point x_i and x_i'

Note: $x_i' F x_i = 0$

Answer 1 *If we let* $\mathbf{x}_i = \begin{bmatrix} x_i & y_i & 1 \end{bmatrix}^T$ *and* $\mathbf{x}_i' = \begin{bmatrix} x_i' & y_i' & 1 \end{bmatrix}^T$, *by expanding* (1) *we obtain:*

$$(x_i' x_i) f_1 + (y_i' x_i) f_2 + x_i f_3 + (x_i' y_i) f_4 + (y_i' y_i) f_5 + y_i f_6 + x_i' f_7 + y_i' f_8 + f_9 = 0$$

that, in vector form can be written as:

$$\begin{bmatrix} x_i' x_i & y_i' x_i & x_i & x_i' y_i & y_i' y_i & y_i & x_i' & y_i' & 1 \end{bmatrix} \boldsymbol{f}$$

Hence we conclude that:

$$\boldsymbol{a}(\boldsymbol{x}_i, \boldsymbol{x}_i') = \begin{bmatrix} x_i' x_i & y_i' x_i & x_i & x_i' y_i & y_i' y_i & y_i & x_i' & y_i' & 1 \end{bmatrix}$$

7.7 Question 7: Fundamental Matrix Details

Answer the following questions, providing an adequate justification.

1. How many degrees of freedom does the fundamental matrix F have?
2. How many independent equations do we need to estimate f? Why?
3. Explain how the system of equations can be written in matrix form as Af = 0. What are the dimensions of A? Write explicitly A.
4. How would you estimate f via constrained least squares?

Answer 2 *Note how most of the considerations are very similar to those we made in order to estimate an homography.*

- *Since the constraint* (1) *holds in the projective space, F is defined up to a scale factor and therefore there are only 8 independent ratios. Moreover the rank of the matrix is 2 (i.e. the determinant is zero). Hence the fundamental matrix has 7 degrees of freedom.*

- *Since F has 7 degrees of freedom we need at least 7 points (plus two extra constraints that will fix the scale and enforce the rank condition). Note that the algorithm that we are developing is called 8 points algorithm and it uses at least 8 points, since the rank condition is not enforced explicitly.*
- *The set of equations for N point pairs can be written in matrix form as:*

$$A\boldsymbol{f} = \begin{bmatrix} \boldsymbol{a}(\boldsymbol{x}_1, \boldsymbol{x}'_1) \\ \boldsymbol{a}(\boldsymbol{x}_2, \boldsymbol{x}'_2) \\ \vdots \\ \boldsymbol{a}(\boldsymbol{x}_N, \boldsymbol{x}'_N) \end{bmatrix} \boldsymbol{f} = 0$$

  where $A \in \mathbb{R}^{N\times 9}$.
- *Similarly to what we did for homographies, we need to solve the constrained least square problem:*

$$\hat{\boldsymbol{f}} = \underset{\boldsymbol{f}\in\mathbb{R}^9, \|\boldsymbol{f}\|=1}{\operatorname{argmin}} \|A\boldsymbol{f}\|$$

7.8 Question 8: One more Time - Fundamental Matrix

Recall that the fundamental matrix can be written as $F = \hat{e'}K'RK^{-1}$. Show that $e' \times x' = Fx$.

Answer 4:

$$
\begin{aligned}
F\mathbf{x} &= [\mathbf{e}']_\times K'RK^{-1}\mathbf{x} \\
&= (\text{since } \mathbf{x} = P_1\mathbf{X}) \\
&= [\mathbf{e}']_\times K'RK^{-1}K \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \mathbf{X} \\
&= [\mathbf{e}']_\times K'R \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \mathbf{X} \\
&= [\mathbf{e}']_\times K' \begin{bmatrix} R & \mathbf{0} \end{bmatrix} \mathbf{X}
\end{aligned}
$$

We are almost there. Following the suggestion we recall that in Question 2 we showed that $\mathbf{e}' = K'\mathbf{t}$ and that the cross product of a vector with itself is zero. Therefore we have that:

$$
\begin{aligned}
F\mathbf{x} &= [\mathbf{e}']_\times K' \begin{bmatrix} R & \mathbf{0} \end{bmatrix} \mathbf{X} \\
&= [\mathbf{e}']_\times K' \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \mathbf{X} \\
&= [\mathbf{e}']_\times \mathbf{x}' \\
&= \mathbf{e}' \times \mathbf{x}'
\end{aligned}
$$