Additional Notes on Velocity Twist Coordinates

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1 Algebraic Interpretation of Velocity Twist Coordinates

Given a spatial frame A and a body frame B, there is a time-dependent transformation between them:

$$g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix} \tag{1}$$

We saw that we can express both the spatial and body velocity twist coordinates in terms of the R_{ab} and p_{ab} (the components of g_{ab}).

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \begin{bmatrix} -\dot{R}_{ab} R^{-1}_{ab} p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R^{-1}_{ab})^\vee \end{bmatrix} \tag{2}$$

$$V^b_{ab} = \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix} = \begin{bmatrix} R^{-1}_{ab} \dot{p}_{ab} \\ (R^{-1}_{ab} \dot{R}_{ab})^\vee \end{bmatrix} \tag{3}$$

The textbook claims that each of these entries corresponds to a physical interpretation as summarized in the following table:

Quantity	Interpretation
ω^s_{ab}	Angular velocity of B wrt frame A, viewed from A.
v^s_{ab}	Velocity of a (possible imaginary) point attached to B traveling through the origin of A wrt A, viewed from A.
ω^b_{ab}	Angular velocity of B wrt frame A, viewed from B.
v^b_{ab}	Velocity of origin of B wrt frame A, viewed from B.

To convince ourselves that this is true, let's come up with algebraic expressions for each interpretation, and see that they match what is proposed in Eqs. 2 and 3.

$\omega^{\mathbf{s}}_{\mathbf{ab}}$: Angular velocity of B wrt frame A, viewed from A.
When measuring the angular velocity of a frame, we need to look only at how its orientation evolves. From the perspective of spatial frame A, the angular velocity of B is the same no matter where it is in space. Thus, we can ignore the $p_{ab}(t)$ component and treat B as if its origin was stuck to A's origin (ie. a pure rotation case).

The angular velocity of a rigid body is the same for *any* point on the body. Thus, let's choose an arbitrary point q attached to the moving body B and try to find its angular velocity with respect to frame A. Recall from the textbook that for a pure rotation case and a point q attached to B,

$$\dot{q}_a(t) = \dot{R}_{ab}(t) R^{-1}_{ab}(t) q_a(t) \tag{4}$$

$$\dot{q}_a(t) = (\dot{R}_{ab}(t) R^{-1}_{ab}(t))^\vee \times q_a(t) \tag{5}$$

So, the angular velocity of B with respect to A, and viewed from A is simply $\boxed{\omega^s_{ab} = (\dot{R}_{ab}R^{-1}_{ab})^\vee}$.

$\omega^{\mathbf{b}}_{\mathbf{ab}}$: Angular velocity of B wrt frame A, viewed from B.
ω^b_{ab} is just ω^s_{ab} expressed in the B frame, so

$$\omega^b_{ab} = R^{-1}_{ab}\omega^s_{ab} \tag{6}$$

To rewrite the above expression, we use the following fact that holds for any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$:

$$\widehat{R\omega} = R\widehat{\omega}R^{-1} \tag{7}$$

Thus,

$$\begin{aligned}
\widehat{\omega}^b_{ab} &= \widehat{R^{-1}_{ab}\omega^s_{ab}} && (8)\\
&= R^{-1}_{ab}\widehat{\omega}^s_{ab}R_{ab} && (9)\\
&= R^{-1}_{ab}\dot{R}_{ab}R^{-1}_{ab}R_{ab} && (10)\\
&= R^{-1}_{ab}\dot{R}_{ab} && (11)
\end{aligned}$$

So $\boxed{\omega^b_{ab} = (R^{-1}_{ab}\dot{R}_{ab})^\vee}$.

$\mathbf{v}^{\mathbf{b}}_{\mathbf{ab}}$: Velocity of origin of B wrt frame A, viewed from B.
The velocity of B's origin with respect to A and viewed from A is $\dot{p}_{ab}$. To express this velocity vector in B coordinates, we simply apply the appropriate rigid body transformation. Recall that only the rotation component of the transformation matters when transforming vectors, so $\boxed{v^b_{ab} = R^{-1}_{ab}\dot{p}_{ab}}$

$\mathbf{v}^{\mathbf{s}}_{\mathbf{ab}}$: Velocity of a (possibly imaginary) point attached to B traveling through the origin of A wrt A, viewed from A.
There exists *some* point, let's call it q, that is attached to B and is instantaneously passing through the origin of A. Its coordinates in the B frame, q_b, are found by applying the rigid body transformation from frame A to the origin in the A frame.

$$q_b = g^{-1}_{ab}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}-R^{-1}_{ab}p_{ab}\\1\end{bmatrix} \tag{12}$$

where q_b is a constant because q is attached to B. Then, the velocity of this point with respect to frame A is

$$\dot{p}_a = \dot{g}_{ab}p_b = \begin{bmatrix}\dot{R}_{ab} & \dot{p}_{ab}\\0 & 0\end{bmatrix}\begin{bmatrix}-R^{-1}_{ab}p_{ab}\\1\end{bmatrix} = \begin{bmatrix}-\dot{R}_{ab}R^{-1}_{ab}p_{ab} + \dot{p}_{ab}\\0\end{bmatrix} \tag{13}$$

As a 3-vector, we have $\boxed{v^s_{ab} = -\dot{R}_{ab}R^{-1}_{ab}p_{ab} + \dot{p}_{ab}}$

2 The Circle Method for Finding Velocity Twist Coordinates

The "circle method" is a convenient way to quickly read off v^s_{ab} and v^b_{ab} if there is a revolute joint that the body frame is attached to:

- Freeze A and B (it doesn't matter where they are).
- Draw a circle perpendicular to the joint's rotation axis, centered at the joint axis, that passes through the origin of A. Imagine a particle moving along this circle with angular speed $\dot{\theta}$. The particle's velocity as it passes through A, with respect to A and viewed from A, is v^s_{ab}.
- Draw a circle perpendicular to the joint's rotation axis, centered at the joint axis, that passes through the origin of B. Imagine a particle moving along this circle with angular speed $\dot{\theta}$. The particle's velocity as it passes through B, with respect to B and viewed from B, is v^b_{ab}.

Let's look at an example:

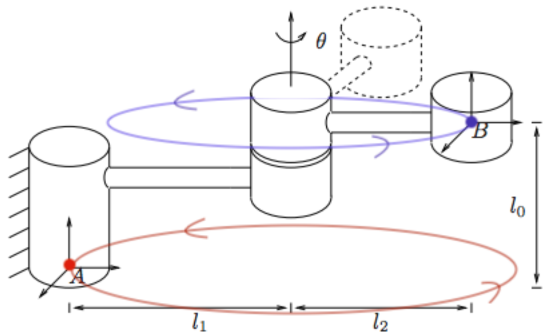

Figure 1: Rigid body motion by rotation about one joint

The velocity of the particle passing through the origin of A, with respect to and expressed in A, is in the positive x direction with magnitude $l_1\dot{\theta}$. Thus, $v^s_{ab} = \begin{bmatrix} l_1\dot{\theta} \\ 0 \\ 0 \end{bmatrix}$

The velocity of the particle passing through the origin of B, with respect to and expressed in B, is in the negative x direction with magnitude $l_2\dot{\theta}$. Thus, $v^b_{ab} = \begin{bmatrix} -l_2\dot{\theta} \\ 0 \\ 0 \end{bmatrix}$