Problem 1. Jacobian for a 4DOF manipulator

Figure 1 shows a 4DOF manipulator with 3 revolute joints and 1 prismatic joint (joint 3) in its initial configuration $\theta = 0$.

(a) Compute the spatial Jacobian $J^s$ and the body Jacobian $J^b$ of the manipulator in the configuration shown.

(b) Now let the robot move so that $\theta_2 = \pi/2$, with all other joints remaining at zero. Compute the spatial Jacobian $J^s$ and body Jacobian $J^b$ in the new configuration.

(c) In which configurations, the one in part (a), or the one in part (b), is the robot in a singular configuration? Justify.
(d) During the execution of a smooth joint trajectory $\theta(t) \in \mathbb{R}^4$, the robot passes through the configuration from part (a) with joint velocities $\dot{\theta}(t) = (0, -1/L, 1, 1/L)$. Find the velocity of the origin of the end effector as seen from the spatial frame at that instant. Note that here we are asking for the velocity of the point at the origin of the tool frame, so your answer should just be a vector $\dot{p}_{sb} \in \mathbb{R}^3$.

Problem 2. Singularities of Euler Angles

(a) Write down the adjoint of a rotation about the origin by rotation matrix $R$.

(b) Using this, demonstrate that the singularity of ZYX (extrinsic) Euler angles occurs when $\theta_2 = \frac{\pi}{2}$. Intuitively, why does this occur?

*Hint: You should model these Euler angles as the product of three revolute joints along the $X, Y, Z$ axes and show that when the second angle is $\pi/2$ the Jacobian loses rank.*

(c) Prove that any rotation represented by three rotations about arbitrary axes ($R = e^{\hat{\omega} \theta_1} e^{\hat{\omega} \theta_2} e^{\hat{\omega} \theta_3}$) will have a singularity.

*Hint 1: First prove the case where $a \omega_1 + b \omega_2 = \omega_3$ (such as with ZYZ Euler angles) i.e. when the three axes are linearly dependent. Then prove the harder case where $\omega_1, \omega_2, \omega_3$ are linearly independent.

*Hint 2: For the second part, finding a set of angles in terms of $\omega_1, \omega_2, \omega_3$ that will always produce a singular configuration is sufficient. Try drawing it out!*

Problem 3. Kinematic Singularity: prismatic joint perpendicular to two parallel coplanar revolute joints

A prismatic joint with twist $\xi_3 = (v_3, 0)$ is normal to a plane containing two parallel revolute axes $\xi_i = (q_i \times \omega_i, \omega_i), \ i = 1, 2$ if

- $v_3^T \omega_i = 0$
- $v_3^T (q_1 - q_2) = 0$
- $\omega_1 = \pm \omega_2$

Show that when this occurs, any six degree of freedom manipulator is at a singular configuration. Give an example of a manipulator exhibiting such a singularity.