Homework 6

EECS/BioE/MechE C106A/206A
Introduction to Robotics

Due: October 19, 2021

Problem 1. Properties of the Adjoint

The Adjoint transformation associated with $g \in SE(3)$ is a 6×6 matrix Ad_g and is defined in equation (2.58) of MLS. Further recall that for any $\xi \in \mathbb{R}^6$, we have $\mathrm{Ad}_g\,\xi = (g \cdot \hat{\xi} \cdot g^{-1})^\vee$.

(a) Show that $(\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}}$ for all $g \in SE(3)$.

(b) Show that $\mathrm{Ad}_{g_1 g_2} = \mathrm{Ad}_{g_1}\mathrm{Ad}_{g_2}$ for all $g_1, g_2 \in SE(3)$.

(c) Prove MLS Proposition 2.15: $V^b_{ac} = \mathrm{Ad}_{g_{bc}^{-1}} V^b_{ab} + V^b_{bc}$.

Hint: It may help to take the "hat" of both sides first.

Problem 2. Twists as Velocities

Recall that Chasle's theorem allows us to express any rigid body transform $g \in SE(3)$ as the exponential of a (not necessarily unit) twist $\xi \in \mathfrak{se}(3)$ as $g = e^{\hat{\xi}}$. Consider a trajectory $g(t) \in SE(3)$ that evolves from an initial configuration $g(0)$ to a final configuration $g(1)$ according to the following formula:

$$g(t) = e^{\hat{\xi} t} \cdot g(0) \tag{1}$$

where $t \in [0, 1]$ denotes time. $g(t)$ evolves according to a *constant screw motion.*

(a) Given a desired initial configuration $g_0 \in SE(3)$ and a desired final configuration $g_1 \in SE(3)$, how might we find a twist $\xi \in \mathfrak{se}(3)$ so that a smooth trajectory of the form (1) takes us from $g(0) = g_0$ to $g(1) = g_1$? You do not need to explicitly solve for the twist, just state the theorem that allows us to guarantee the existence of such a ξ.

(b) For $t \in (0, 1)$, find the spatial rigid body velocity $V^s(t)$ of the trajectory you constructed in part (a). Does this velocity depend on time?

(c) For $t \in (0, 1)$, find the body velocity $V^b(t)$ of the trajectory you constructed in part (a). Does this velocity depend on time?

(d) Let a $g \in SE(3)$ be given, with exponential coordinates ξ (a not necessarily unit twist) so that $g = e^{\hat{\xi}}$. Interpret the twist ξ as a rigid body velocity that, when performed uniformly for 1 second, brings a rigid body from the identity configuration to the configuration g. In this way, interpret twists (and the idea of exponential coordinates) in terms of rigid body velocities.

Problem 3. Velocities as Twists

Consider a smooth rotational trajectory $R(t) \in SO(3)$ where $t \in [0, \infty)$ denotes time. In this problem, we will derive the notion of angular and rigid body velocities directly from our knowledge of exponential coordinates.

(a) Let $t \in [0, \infty)$ and a small $\Delta t > 0$ be given. Argue that there exists $\hat{\omega} \in \mathfrak{so}(3)$ such that

$$R(t + \Delta t) = e^{\hat{\omega}\Delta t} \cdot R(t) \tag{2}$$

Note that ω is a function of both t and Δt.

(b) Now take the limit as $\Delta t \to 0$. Show that in this limit, $\hat{\omega}$ approaches $\dot{R}R^T = \hat{\omega}^s(t)$. That is, in the limit, this infinitessimal rotation approaches the spatial angular velocity of R. *Hint: It may help to recall that for small Δt we have $e^{A\Delta t} \approx I + A\Delta t$. It may also help to recall the limit definition of the derivative.*

(c) Conclude that the spatial angular velocity of R is simply the *instantaneous rotation axis* of the body, with magnitude equal to the instantaneous angular speed.

(d) Repeat the exercise in parts (a)-(c) except with a smooth *rigid-body motion* trajectory $g(t) \in SE(3)$. Interpret the spatial velocity $V^s(t)$ in terms of the twist associated with the *instantaneous screw motion* that the body is undergoing at time t.