

EECS/BioE/MechE C106A Discussion 8: Jacobians

1 Overview

Last week, we learned about spatial and body velocity twists between two frames A and B . These velocity twists are useful because they allow us to find the instantaneous velocity of the B frame expressed in both spatial and body coordinates.

$$v_{q_a}(t) := \dot{q}_a(t) = \dot{g}_{ab}(t)q_b = \underbrace{\dot{g}_{ab}(t)g_{ab}^{-1}(t)}_{:=\hat{V}_{ab}^s} q_a = \hat{V}_{ab}^s q_a \quad (1)$$

$$v_{q_b}(t) := g_{ab}^{-1}(t)v_{q_a}(t) = \underbrace{g_{ab}^{-1}(t)\dot{g}_{ab}(t)}_{:=\hat{V}_{ab}^b} q_b = \hat{V}_{ab}^b q_b \quad (2)$$

Today, we will be thinking of velocities in the context of robotic manipulators. We will be finding the velocities between the fixed frame S and the end effector frame T , \hat{V}_{st}^s and \hat{V}_{st}^b .

To do so, we will introduce the notion of spatial and body manipulator Jacobians. Then, we will see how these manipulator Jacobians help us detect singular configurations.

2 Adjoint for Twist Coordinate Change

When working with twists, we can transform a twist matrix $\hat{\xi}$ into a different coordinate system defined by g , so that it becomes $\hat{\xi}'$

$$\hat{\xi}' = g\hat{\xi}g^{-1} \quad (3)$$

In twist coordinates,

$$\xi' = Ad_g \xi \quad (4)$$

3 Spatial Jacobians

3.1 Definition

As before, we have the expression for \widehat{V}_{st}^s as a function of the transformation between S and T :

$$\widehat{V}_{st}^s = \dot{g}_{st}(\theta)g_{st}^{-1}(\theta) \quad (5)$$

In twist coordinates,

$$V_{st}^s = J_{st}^s(\theta)\dot{\theta} \quad (6)$$

where the *spatial manipulator Jacobian* $J_{st}^s(\theta)$ is defined as

$$J_{st}^s(\theta) = \begin{bmatrix} (\frac{\partial g_{st}}{\partial \theta_1})^\vee & \dots & (\frac{\partial g_{st}}{\partial \theta_n})^\vee \end{bmatrix} \quad (7)$$

$$= [\xi_1 \quad \xi'_2 \quad \dots \quad \xi'_n] \quad (8)$$

$$\xi'_i = Ad_{(e^{\widehat{\xi}_1 \theta_1} \dots e^{\widehat{\xi}_{i-1} \theta_{i-1}})} \xi_i \quad (9)$$

3.2 Interpretation

For some configuration θ , the spatial manipulator Jacobian maps the joint velocity vector $\dot{\theta}$ into the spatial velocity twist coordinates of the end-effector.

The i^{th} column of the spatial Jacobian ξ'_i is equal to the i^{th} joint twist transformed to the current manipulator configuration and written in spatial coordinates.

Problem 1. *Explain how this physical interpretation is true.*

3.3 How it's used

We can use the spatial Jacobian to compute the instantaneous velocity of a point q attached to the end-effector relative to the spatial frame. This velocity is

$$v_{q_s} = \widehat{V}_{st}^s q_s = (J_{st}^s(\theta)\dot{\theta})^\wedge q_s \quad (10)$$

where q_s is the coordinates of q in the spatial frame.

4 Body Jacobians

4.1 Definition

Now let's look at velocity twists in the body frame rather than in the spatial frame:

$$\widehat{V}_{st}^b = g_{st}^{-1}(\theta) \dot{g}_{st}(\theta) \quad (11)$$

In twist coordinates,

$$V_{st}^b = J_{st}^b(\theta) \dot{\theta} \quad (12)$$

where the *body manipulator Jacobian* $J_{st}^b(\theta)$ is defined as

$$J_{st}^b(\theta) = [\xi_1^\dagger \quad \xi_2^\dagger \quad \dots \quad \xi_n^\dagger] \quad (13)$$

$$\xi_i^\dagger = Ad_{(e^{\widehat{\xi}_{i+1}} \theta_{i+1} \dots e^{\widehat{\xi}_n} \theta_n g_{st}(0))}^{-1} \xi_i \quad (14)$$

4.2 Interpretation

For some configuration θ , the body manipulator Jacobian maps the joint velocity vector $\dot{\theta}$ into the body velocity twist coordinates of the end-effector.

The i^{th} column of the body Jacobian ξ_i^\dagger is equal to the i^{th} joint twist transformed to the current manipulator configuration and written in body coordinates.

Problem 2. *Explain how this physical interpretation is true.*

4.3 How it's used

We can use the body Jacobian to compute the instantaneous velocity of a point q attached to the end-effector relative to the body frame. This velocity is

$$v_{q_b} = \widehat{V}_{st}^b q_b = (J_{st}^b(\theta) \dot{\theta})^\wedge q_b \quad (15)$$

where q_b is the coordinates of q in the tool frame.

4.4 Converting between Spatial and Body Jacobians

$$J_{st}^s(\theta) = Ad_{g_{st}(\theta)} J_{st}^b(\theta) \quad (16)$$

Problem 3. Find the spatial and body manipulator Jacobians for the Stanford manipulator.

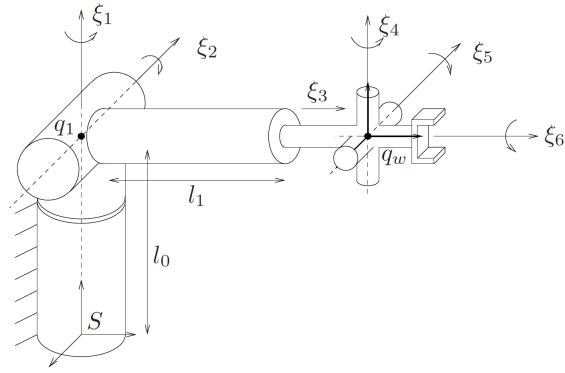
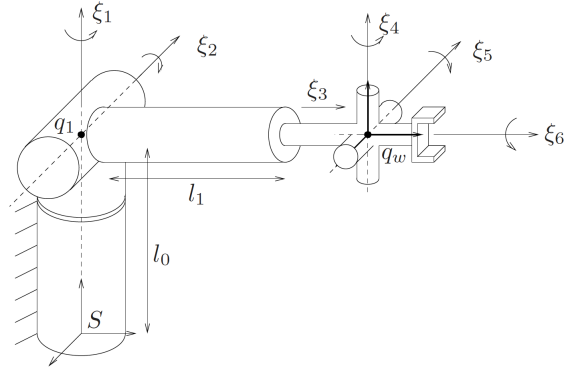


Figure 1: Stanford manipulator

When we want to find the manipulator Jacobians for some specific configuration θ_d , it's easier to do it by inspection rather than having to first find the manipulator Jacobians for general θ , then plugging in θ_d . To find cross products, it may be helpful to draw out circles to visualize direction.

Problem 4. Find the spatial and body manipulator Jacobians for the Stanford manipulator in its initial configuration. In this case, $\theta_d = 0$.



5 Singularities

$$V_{st}^s = J_{st}^s(\theta)\dot{\theta}$$

At some configuration θ_s , it may be possible for $J_{st}^s(\theta_s)$ to *not* have full rank. In this case, $J_{st}^s(\theta_s)$ is not invertible, and thus the manipulator is unable to achieve instantaneous motion in certain directions. We call θ_s a *singular configuration*. Since being in singular configurations is not desirable, it's important to figure out what they are for a particular manipulator so they can be avoided.

Problem 5. *Show that a manipulator Jacobian is singular if there exist four revolute joint axes that intersect.*

Problem 6. *When is the elbow manipulator in a singular configuration?*

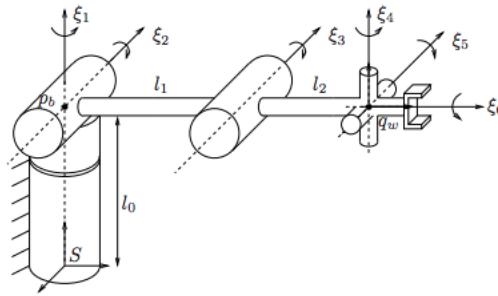


Figure 2: Elbow manipulator