EE106A Discussion 7: Velocities and Adjoints

1 Representing Velocities

How do we express the velocity of something, for example, a point q? The velocity is the rate of change of its position with *with respect to a reference frame*. Thus, if we have a frame called U, the velocity of the point q with respect to U is $\dot{q}_u(t)$, where $q_u(t)$ is the point's position with respect to U as a function of time. Now we have a time-dependent velocity vector $\dot{q}_u(t)$ in frame U.

However, this velocity vector can be viewed from any other frame V, and can therefore be expressed from V. Recall that we can change the frame of reference if we have transformation g_{uv}. When representing velocities, it's important to keep in mind that there are two (often different) frames that are relevant.

2 Rigid Body Velocities

Let's say we have a fixed frame A and a moving frame B. By construction, let's also have point q that's attached to frame B. We call A the *spatial* coordinate frame, and B the *body* coordinate frame.

2.1 Spatial Velocity

Since frame B is moving, the transformation between A and B is time-dependent:

$$g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix} \tag{1}$$

Because q is fixed to frame B, its coordinates with respect to B, q_b, are constant. Its coordinates in frame A, however, are time-dependent:

$$q_a(t) = g_{ab}(t) q_b \tag{2}$$

Now, the velocity of this point with respect to A, and also viewed from A, is found via differentiation:

$$\dot{q}_a(t) = \dot{g}_{ab}(t) q_b \tag{3}$$

Following the textbook, we define this quantity to be $v_{q_a}(t) \coloneqq \dot{q}_a(t)$. We can further express q_b in terms of $q_a(t)$ by the appropriate transformation:

$$\boxed{v_{q_a}(t) \coloneqq \dot{q}_a(t) = \dot{g}_{ab}(t) q_b = \underbrace{\dot{g}_{ab}(t) g_{ab}^{-1}(t)}_{\coloneqq \widehat{V}^s_{ab}} q_a = \widehat{V}^s_{ab} q_a} \tag{4}$$

It turns out that $\dot{g}_{ab}(t) g_{ab}^{-1}(t)$ is a skew symmetric matrix, and we define it to be the *spatial velocity* $\widehat{V}^s_{ab}$. Notice that this spatial velocity is a twist.

Going forward with the algebra and dropping notation for time-dependence, we have

$$\widehat{V}^s_{ab} \coloneqq \dot{g}_{ab} g^{-1}_{ab} = \begin{bmatrix} \dot{R}_{ab} R^T_{ab} & -\dot{R}_{ab} R^T_{ab} p_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \tag{5}$$

and applying the "vee" operator, we get the twist coordinates

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \begin{bmatrix} -\dot{R}_{ab} R^T_{ab} p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R^T_{ab})^\vee \end{bmatrix} \tag{6}$$

2.2 Body Velocity

We found the the velocity v_{q_a}, and now let's find v_{q_b}. This v_{q_b} is NOT the velocity of point q relative to B and viewed from B — that would always be zero. Rather, it is the velocity of q relative to A and viewed from B, which is related to v_{q_a} by a transformation:

$$v_{q_b}(t) = g^{-1}_{ab}(t) v_{q_a}(t) \tag{7}$$

Again, the notation in the textbook is a bit misleading, with $v_{q_a}(t) \coloneqq \dot{q}_a(t)$, but $v_{q_b}(t) \neq \dot{q}_b = 0$.

We can also find a *body velocity* $\widehat{V}^b_{ab}$ (which is also a twist) such that

$$\boxed{v_{q_b}(t) = g^{-1}_{ab}(t) v_{q_a}(t) = \underbrace{g^{-1}_{ab}(t)\dot{g}_{ab}}_{\coloneqq \widehat{V}^b_{ab}} q_b = \widehat{V}^b_{ab} q_b} \tag{8}$$

Again, dropping the time dependency,

$$\widehat{V}^b_{ab} \coloneqq g^{-1}_{ab}(t)\dot{g}_{ab} = \begin{bmatrix} R^T_{ab}\dot{R}_{ab} & R^T_{ab}\dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \tag{9}$$

and the twist coordinates are

$$V^b_{ab} = \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix} = \begin{bmatrix} R^T_{ab}\dot{p}_{ab} \\ (R^T_{ab}\dot{R}_{ab})^\vee \end{bmatrix} \tag{10}$$

3 Example: One DOF Manipulator

Problem 1. *Find the body and spatial velocities for the fixed frame A and moving frame B.*

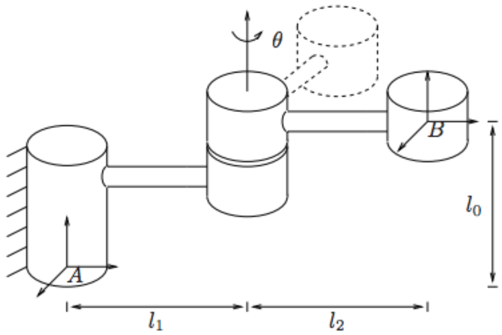


Figure 1: Rigid body motion by rotation about one joint

We can find the spatial and body velocities by first finding the transformation between the two frames $g_{ab}(t)$. If we go forth with the forward kinematics, we get

$$g_{ab}(t) = \begin{bmatrix} cos\theta(t) & -sin\theta(t) & 0 & -l_2 sin\theta(t) \\ sin\theta(t) & cos\theta(t) & 0 & l_1 + l_2 cos\theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From this transformation, we calculate the derivative and the inverse:

$$\dot{g}_{ab} = \begin{bmatrix} -sin\theta & -cos\theta & 0 & -l_2 cos\theta \\ cos\theta & -sin\theta & 0 & -l_2 sin\theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

$$g_{ab}^{-1} = \begin{bmatrix} cos\theta & sin\theta & 0 & -l_1 sin\theta \\ -sin\theta & cos\theta & 0 & -l_2 - l_1 cos\theta \\ 0 & 0 & 1 & -l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\widehat{V}_{ab}^s = \dot{g}_{ab} g_{ab}^{-1} = \begin{bmatrix} 0 & -1 & 0 & l_1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\theta} \implies V_{ab}^s = \begin{bmatrix} l_1\dot{\theta} \\ 0 \\ 0 \\ 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

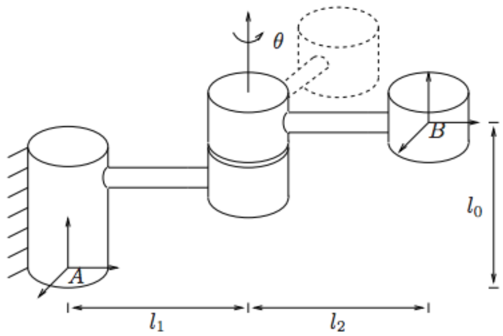
$$\widehat{V}_{ab}^b = \dot{g}_{ab} g_{ab}^{-1} = \begin{bmatrix} 0 & -1 & 0 & -l_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\theta} \implies V_{ab}^b = \begin{bmatrix} -l_2\dot{\theta} \\ 0 \\ 0 \\ 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

There's actually a shorter method we can use to find the velocities. It's based on the interpretation of the twist coordinates as summarized in the following table:

Quantity	Interpretation
ω^s_{ab}	Angular velocity of B wrt frame A, viewed from A.
v^s_{ab}	Velocity of a (possible imaginary) point attached to B traveling through the origin of A wrt A, viewed from A.
ω^b_{ab}	Angular velocity of B wrt frame A, viewed from B.
v^b_{ab}	Velocity of origin of B wrt frame A, viewed from B.

When the velocities are induced by revolute joints, we can imagine circular paths traced out by these joints that help us figure out these values. Let's see how this works with some examples:

Problem 2. *Find the spatial and body velocity twists for the fixed frame A and moving frame B in Fig. 1 (copied here) using the interpretation of twist coordinates.*



$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix}; V^b_{ab} = \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix}$$

ω^s_{ab}: With respect to A and expressed in A-coordinates, the angular velocity of B is $\begin{bmatrix} 0 & 0 & \dot{\theta} \end{bmatrix}^T$.

ω^b_{ab}: With respect to A and expressed in B-coordinates, the angular velocity of B is also $\begin{bmatrix} 0 & 0 & \dot{\theta} \end{bmatrix}^T$.

v^s_{ab}: A point attached to B that travels through the origin of A has a circular trajectory around the joint axis that passes through the origin of A. Instantaneously at the origin of A, with respect to A and in A-coordinates, the velocity of such a point is $\begin{bmatrix} l_1\dot{\theta} & 0 & 0 \end{bmatrix}^T$.

v^b_{ab}: The velocity of the origin of B with respect to A, would be tangential to the circle that the origin of B traces as a function of θ. Thus, as expressed in B-coordinates, this velocity is along the negative x-axis, with coordinates $\begin{bmatrix} -l_2\dot{\theta} & 0 & 0 \end{bmatrix}^T$.

Problem 3. *Find the spatial and body velocity twists between A to B and also between B to C in Fig. 2 using the interpretation of twist coordinates.*

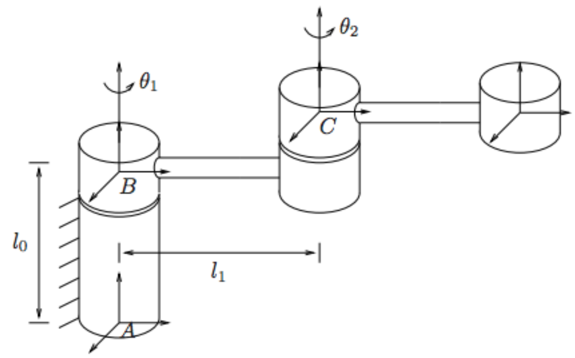


Figure 2: Rigid body motion by rotation about two joints.

$$V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix}; V^b_{ab} = \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix}; V^s_{bc} = \begin{bmatrix} v^s_{bc} \\ \omega^s_{bc} \end{bmatrix}; V^b_{bc} = \begin{bmatrix} v^b_{bc} \\ \omega^b_{bc} \end{bmatrix}$$

ω^s_{ab}: With respect to A and expressed in A-coordinates, the angular velocity of B is $\begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix}^T$.

ω^b_{ab}: With respect to A and expressed in B-coordinates, the angular velocity of B is also $\begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix}^T$.

v^s_{ab}: A point attached to B that travels through the origin of A will rotate about the axis of joint 1. However, since the origin of A is also on this axis, this point will not move with respect to A. Thus, its velocity is $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$.

v^b_{ab}: The velocity of the origin of B with respect to A would also be zero, $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$.

ω^s_{bc}: With respect to B and expressed in B-coordinates, the angular velocity of C is $\begin{bmatrix} 0 & 0 & \dot{\theta}_2 \end{bmatrix}^T$.

ω^b_{bc}: With respect to B and expressed in C-coordinates, the angular velocity of C is also $\begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix}^T$.

v^s_{bc}: A point attached to C that travels through the origin of B will rotate about the axis of joint 2. With respect to frame B, its instantaneous velocity at its origin is $\begin{bmatrix} l_1\dot{\theta} & 0 & 0 \end{bmatrix}^T$.

v^b_{bc}: The velocity of the origin of C with respect to B is zero, $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$.

4 Adjoint Transformations

The spatial and body velocity of a rigid motion are related by a similarity transform:

$$\widehat{V}^s_{ab} = g_{ab}\widehat{V}^b_{ab}g^{-1}_{ab} \tag{11}$$

and

$$\boxed{V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & \widehat{p}_{ab}R_{ab} \\ 0 & R_{ab} \end{bmatrix}}_{:=Ad_{g_{ab}}} \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix} = Ad_{g_{ab}}V^b_{ab}} \tag{12}$$

where the *adjoint transformation* $Ad_{g_{ab}}$ maps body velocity twist coordinates to spatial velocity twist coordinates.
Ad_g is invertible, and its inverse is

$$Ad^{-1}_{g_{ab}} = \begin{bmatrix} R^T_{ab} & -R^T_{ab}\widehat{p} \\ 0 & R^T_{ab} \end{bmatrix} \tag{13}$$

4.1 Coordinate Transformations

We don't compose velocities in the same way as with rigid body transformations—we need to use the adjoints. For spatial velocity composition, we have:

$$V^s_{ac} = V^s_{ab} + Ad_{g_{ab}}V^s_{bc} \tag{14}$$

And with body velocity composition,

$$V^b_{ac} = Ad_{g^{-1}_{bc}}V^b_{ab} + V^b_{bc} \tag{15}$$

Extra practice: *Using your solutions to Problem 3 and the appropriate adjoint transformation, find* V^s_{ac} *in Fig. 2. Your answer should match the solution to Example 2.6 on page 60 in MLS.*