

# EE106A Discussion 7: Velocities and Adjoint

## 1 Representing Velocities

How do we express the velocity of something, for example, a point  $q$ ? The velocity is the rate of change of its position with *with respect to a reference frame*. Thus, if we have a frame called  $U$ , the velocity of the point  $q$  with respect to  $U$  is  $\dot{q}_u(t)$ , where  $q_u(t)$  is the point's position with respect to  $U$  as a function of time. Now we have a time-dependent velocity vector  $\dot{q}_u(t)$  in frame  $U$ .

However, this velocity vector can be viewed from any other frame  $V$ , and can therefore be expressed from  $V$ . Recall that we can change the frame of reference if we have transformation  $g_{uv}$ . When representing velocities, it's important to keep in mind that there are two (often different) frames that are relevant.

## 2 Rigid Body Velocities

Let's say we have a fixed frame  $A$  and a moving frame  $B$ . By construction, let's also have point  $q$  that's attached to frame  $B$ . We call  $A$  the *spatial* coordinate frame, and  $B$  the *body* coordinate frame.

### 2.1 Spatial Velocity

Since frame  $B$  is moving, the transformation between  $A$  and  $B$  is time-dependent:

$$g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix} \quad (1)$$

Because  $q$  is fixed to frame  $B$ , its coordinates with respect to  $B$ ,  $q_b$ , are constant. Its coordinates in frame  $A$ , however, are time-dependent:

$$q_a(t) = g_{ab}(t)q_b \quad (2)$$

Now, the velocity of this point with respect to  $A$ , and also viewed from  $A$ , is found via differentiation:

$$\dot{q}_a(t) = \dot{g}_{ab}(t)q_b \quad (3)$$

Following the textbook, we define this quantity to be  $v_{q_a}(t) := \dot{q}_a(t)$ . We can further express  $q_b$  in terms of  $q_a(t)$  by the appropriate transformation:

$$\boxed{v_{q_a}(t) := \dot{q}_a(t) = \dot{g}_{ab}(t)q_b = \underbrace{\dot{g}_{ab}(t)g_{ab}^{-1}(t)}_{:= \hat{V}_{ab}^s} q_a = \hat{V}_{ab}^s q_a} \quad (4)$$

It turns out that  $\dot{g}_{ab}(t)g_{ab}^{-1}(t)$  is a skew symmetric matrix, and we define it to be the *spatial velocity*  $\hat{V}_{ab}^s$ . Notice that this spatial velocity is a twist.

Going forward with the algebra and dropping notation for time-dependence, we have

$$\widehat{V}_{ab}^s := \dot{g}_{ab} g_{ab}^{-1} = \begin{bmatrix} \dot{R}_{ab} R_{ab}^T & -\dot{R}_{ab} R_{ab}^T p_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \quad (5)$$

and applying the "vee" operator, we get the twist coordinates

$$V_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} = \begin{bmatrix} -\dot{R}_{ab} R_{ab}^T p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R_{ab}^T)^\vee \end{bmatrix} \quad (6)$$

## 2.2 Body Velocity

We found the the velocity  $v_{q_a}$ , and now let's find  $v_{q_b}$ . This  $v_{q_b}$  is NOT the velocity of point  $q$  relative to  $B$  and viewed from  $B$  — that would always be zero. Rather, it is the velocity of  $q$  relative to  $A$  and viewed from  $B$ , which is related to  $v_{q_a}$  by a transformation:

$$v_{q_b}(t) = g_{ab}^{-1}(t) v_{q_a}(t) \quad (7)$$

Again, the notation in the textbook is a bit misleading, with  $v_{q_a}(t) := \dot{q}_a(t)$ , but  $v_{q_b}(t) \neq \dot{q}_b = 0$ .

We can also find a *body velocity*  $\widehat{V}_{ab}^b$  (which is also a twist) such that

$$\boxed{v_{q_b}(t) = g_{ab}^{-1}(t) v_{q_a}(t) = \underbrace{g_{ab}^{-1}(t) \dot{g}_{ab}}_{:= \widehat{V}_{ab}^b} q_b = \widehat{V}_{ab}^b q_b} \quad (8)$$

Again, dropping the time dependency,

$$\widehat{V}_{ab}^b := g_{ab}^{-1}(t) \dot{g}_{ab} = \begin{bmatrix} R_{ab}^T \dot{R}_{ab} & R_{ab}^T \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \quad (9)$$

and the twist coordinates are

$$V_{ab}^b = \begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix} = \begin{bmatrix} R_{ab}^T \dot{p}_{ab} \\ (R_{ab}^T \dot{R}_{ab})^\vee \end{bmatrix} \quad (10)$$

### 3 Example: One DOF Manipulator

**Problem 1.** Find the body and spatial velocities for the fixed frame  $A$  and moving frame  $B$ .

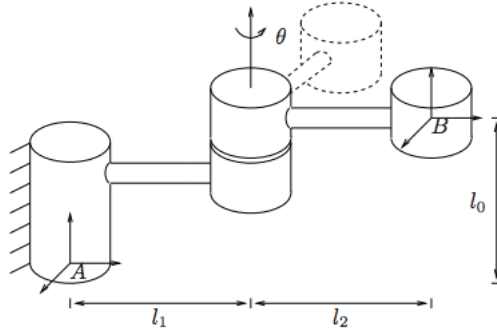


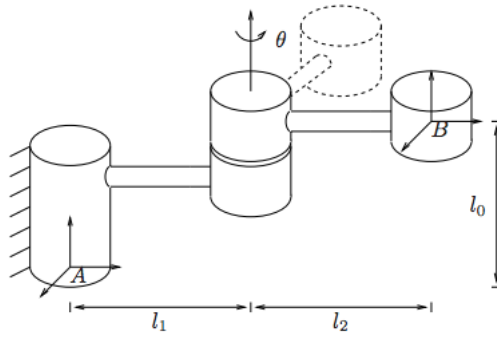
Figure 1: Rigid body motion by rotation about one joint

There's actually a shorter method we can use to find the velocities. It's based on the interpretation of the twist coordinates as summarized in the following table:

Quantity	Interpretation
$\omega_{ab}^s$	Angular velocity of $B$ wrt frame $A$ , viewed from $A$ .
$v_{ab}^s$	Velocity of a (possible imaginary) point attached to $B$ traveling through the origin of $A$ wrt $A$ , viewed from $A$ .
$\omega_{ab}^b$	Angular velocity of $B$ wrt frame $A$ , viewed from $B$ .
$v_{ab}^b$	Velocity of origin of $B$ wrt frame $A$ , viewed from $B$ .

When the velocities are induced by revolute joints, we can imagine circular paths traced out by these joints that help us figure out these values. Let's see how this works with some examples:

**Problem 2.** Find the spatial and body velocity twists for the fixed frame  $A$  and moving frame  $B$  in Fig. 1 (copied here) using the interpretation of twist coordinates.



**Problem 3.** Find the spatial and body velocity twists between  $A$  to  $B$  and also between  $B$  to  $C$  in Fig. 2 using the interpretation of twist coordinates.

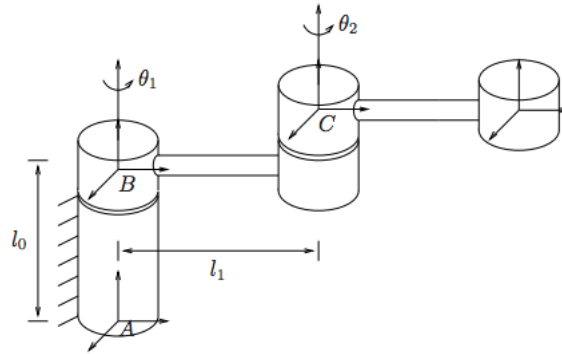


Figure 2: Rigid body motion by rotation about two joints.

## 4 Adjoint Transformations

The spatial and body velocity of a rigid motion are related by a similarity transform:

$$\hat{V}_{ab}^s = g_{ab} \hat{V}_{ab}^b g_{ab}^{-1} \quad (11)$$

and

$$V_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix}}_{:= Ad_{g_{ab}}} \begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix} = Ad_{g_{ab}} V_{ab}^b \quad (12)$$

where the *adjoint transformation*  $Ad_{g_{ab}}$  maps body velocity twist coordinates to spatial velocity twist coordinates.

$Ad_g$  is invertible, and its inverse is

$$Ad_{g_{ab}}^{-1} = \begin{bmatrix} R_{ab}^T & -R_{ab}^T \hat{p} \\ 0 & R_{ab}^T \end{bmatrix} \quad (13)$$

### 4.1 Coordinate Transformations

We don't compose velocities in the same way as with rigid body transformations—we need to use the adjoints. For spatial velocity composition, we have:

$$V_{ac}^s = V_{ab}^s + Ad_{g_{ab}} V_{bc}^s \quad (14)$$

And with body velocity composition,

$$V_{ac}^b = Ad_{g_{bc}}^{-1} V_{ab}^b + V_{bc}^b \quad (15)$$

**Extra practice:** Using your solutions to Problem 3 and the appropriate adjoint transformation, find  $V_{ac}^s$  in Fig. 2. Your answer should match the solution to Example 2.6 on page 60 in *MLS*.