EE106A Discussion 4: Inverse Kinematics

{jaymonga16, amaysaxena, isabella.huang, valmik}@berkeley.edu

1 Inverse kinematics

In forward kinematics, we found the expression for $g_{st}(\theta)$ as a function of θ. Now, in inverse kinematics, we are given a desired configuration of the tool frame g_d, and we wish to find the θ for which

$$e^{\widehat{\xi_1}\theta_1}...e^{\widehat{\xi_n}\theta_n}g_{st}(0) = g_{st}(\theta) = g_d \tag{1}$$

2 Padan-Kahan subproblems

To solve the inverse kinematics problem, one technique is to distill it into the following three simpler subproblems for which we know the solutions.

2.1 Subproblem 1: Rotation about a single axis

Let ξ be a zero-pitch twist along ω with unit magnitude, and p, $q \in \mathbb{R}^3$ be two points. Find θ such that

$$e^{\widehat{\xi}\theta}p = q \tag{2}$$

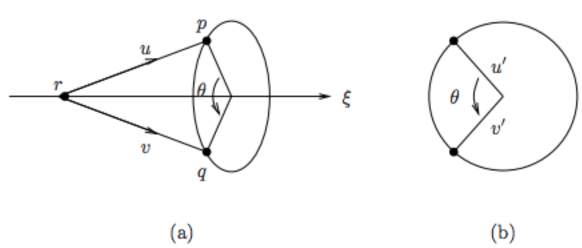

Figure 1: Subproblem 1: a) Rotate p about the axis of ξ until it is coincident with q. b) Projection of u and v onto the plane perpendicular to the twist axis.

- *Define $u = (p - r)$ and $v = (q - r)$ where r is a point on the axis. Find an expression that relates u and v.* $e^{\widehat{\omega}\theta}u = v$

- *Find expressions for u' and v', the projected u and v on the plane perpendicular to the rotation axis.* $u' = u - \omega\omega^T u$ and $v' = v - \omega\omega^T v$

- *Write the necessary conditions for there to be a solution.* $\omega^T u = \omega^T v$ and $||u'|| = ||v'||$

- *Find the solution for θ given that it exists.* By definition of the cross and dot products respectively, $u' \times v' = \omega sin\theta||u'||||v'||$ and $u' \cdot v' = cos\theta||u'||||v'||$. Given that $||\omega|| = 1$, we multiple both sides of the cross product equation by ω^T and divide the two equations to get that $\theta = atan2(\omega^T(u' \times v'), u' \cdot v')$

2.2 Subproblem 2: Rotation about two subsequent axes

Let ξ_1 and ξ_2 be two zero-pitch, unit magnitude twists with intersecting axes, and p, $q \in \mathbb{R}^3$ be two points. Find θ_1 and θ_2 such that

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p = q \tag{3}$$

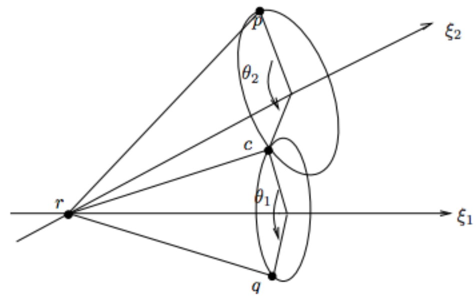

Figure 2: Subproblem 2: Rotate p around the axis of ξ_2, then around the axis of ξ_1 such that the final location is coincident with q.

• *Geometrically, when does there exist zero, one, or multiple solutions to this subproblem?*
When the two circles intersect zero, one, or two times respectively.

Let r be the intersection of the two axes, and c be the intermediate point at which p is rotated about ω_2 by θ_2. Define vectors $u = (p - r)$, $v = (q - r)$, and $z = (c - r)$.

• *Write the expression for z in terms of a transformations applied to u and v.*

$$e^{\widehat{\omega}_2\theta_2}u = z = e^{-\widehat{\omega}_1\theta_1}v$$

Similarly to Subproblem 1, it is true that

$$\omega_2^T u = \omega_2^T z \tag{4}$$

$$\omega_1^T v = \omega_1^T z \tag{5}$$

$$||u|| = ||z|| = ||v|| \tag{6}$$

We can express z as a linear combination of the linearly independent vectors ω_1, ω_2, and $\omega_1 \times \omega_2$:

$$z = \alpha\omega_1 + \beta\omega_2 + \gamma(\omega_1 \times \omega_2) \tag{7}$$

The solutions to these coefficients α, β, and γ are found by using the expressions in Eqs. 4 to 6 (see textbook for full details). There are either zero, one, or two real solutions to these coefficients. If a solution exists, we have z, and hence c.

What's left is to solve

$$e^{\widehat{\xi}_2\theta_2}p = c \tag{8}$$

and

$$e^{-\widehat{\xi}_1\theta_1}q = c \tag{9}$$

which requires us to solve Subproblem 1 twice.

2.3 Subproblem 3: Rotation to a given distance

Let ξ be a zero-pitch, unit magnitude twist, p, $q \in \mathbb{R}^3$ be two points, and $\delta > 0$. Find θ such that:

$$||q - e^{\widehat{\xi}\theta}p|| = \delta \tag{10}$$

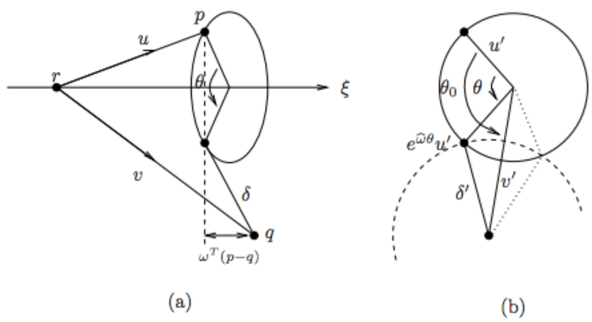

Figure 3: Subproblem 3: a) Rotate p about the axis of ξ until it is a distance δ from point q. b) Projection onto plane perpendicular to axis.

• *Geometrically, when does there exist zero, one, or multiple solutions to this subproblem?*
When the circle formed by p's rotation about ξ intersects the sphere of radius δ with center q zero, one, or two times respectively.

• *Write the expressions of the projected u and v onto the plane perpendicular to ω, which we call u' and v'.*
$u' = u - \omega\omega^T u$ and $v' = v - \omega\omega^T v$

• *Write an expression for the distance δ' (the projected δ onto the plane perpendicular to ω) as a function of u' and v'.*

$$\delta = ||v' - e^{\widehat{\omega}\theta}u'||$$

• *Find the solution for θ. Hint: use result derived in Subproblem 1 to find θ_0.*
Using the same idea as in subproblem 1, we have that

$$\theta_0 = atan2(\omega^T(u' \times v'), u' \cdot v')$$

By the law of cosines,

$$\delta'^2 = ||u'||^2 + ||v'||^2 - 2||u'||||v'||cos(\theta_0 - \theta)$$

so

$$\theta = \theta_0 \pm cos^{-1}\left(\frac{||u'||^2 + ||v'||^2 - \delta'^2}{2||u'||||v'||}\right)$$

3 Using PK subproblems to solve inverse kinematics

We want to simplify complete inverse kinematics problems into the three subproblems we know how to solve. The full equation becomes more simplified when we apply the kinematics equations to *special points*.

In summary, we have the following subproblems that we can use to simplify the inverse kinematics of a robot with revolute joints:

Subproblem 1:	$e^{\widehat{\xi}\theta}p = q$	rotate one point onto another
Subproblem 2:	$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p = q$	rotate about two intersecting axes
Subproblem 3:	$\left\| e^{\widehat{\xi}\theta}p - q \right\| = \delta$	move one point to a specified distance from another

Solving inverse kinematics is a game of trying to reduce the number of unknowns we need to deal with. At each step, we will try to leverage the θ_i's that we know, along with specially chosen points on the manipulator, to reduce the problem to having one or two unknown θ_i's, at which point we can use one of our subproblems to solve for the remaining variables. Let's look at some of the tricks we can use to reduce the number of unknowns we are dealing with.

3.1 Tricks for solving inverse kinematics using PK subproblems

Recall our problem set up. We are given a desired end effector configuration $g_d \in SE(3)$, and we need to find $(\theta_1, ..., \theta_n)$ such that

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0) = g_d$$

It will simplify matters if, whenever possible, we keep all the known matrices on the right hand side and all the unknown matrices on left hand side. So to begin with, we will re-write the above problem to leave only the product of exponentials on the left hand side

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} = g_d g_{st}^{-1}(0) := g$$

where we have simply defined g to be the known matrix $g_d g_{st}^{-1}(0)$ to simplify notation. As we shall see, by picking certain points on our twist axes cleverly, we can eliminate variables from this product of exponentials. We have two primary tricks for this. The first trick will allow us to eliminate exponentials from the right hand side of the expression $e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n}$ and the second trick will allow us to eliminate exponentials from the left and side of that expression.

3.1.1 Trick 1: Apply equations to a point on the axes

If we have a revolute twist ξ and we have a point p on the twist axis, applying the transformation on that point does nothing to it, ie:

$$e^{\widehat{\xi}\theta}p = p \tag{11}$$

For example, if our IK problem is

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}e^{\widehat{\xi}_3\theta_3} = g \tag{12}$$

then choosing a point p on the axis of ξ_3 and multiplying both sides of (12) with p yields

$$\begin{aligned} & e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}e^{\widehat{\xi}_3\theta_3}p = gp \\ \implies & e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}\left(e^{\widehat{\xi}_3\theta_3}p\right) = gp \\ \implies & e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p = gp \end{aligned}$$

and this is simply Subproblem 2. In this way, we have managed to eliminate $e^{\hat{\xi}_3}$ from our equation, allowing us to solve for θ_1 and θ_2 using subproblem 2. Once we know θ_1 and θ_2, now the matrices $e^{\hat{\xi}_1\theta_1}$ and $e^{\hat{\xi}_2\theta_2}$ are both known, and we can return to finding the rest of the θ_i's.

3.1.2 Trick 2: Subtract a point from both sides and take the norm

Remember that rigid motions preserve norm. For example, say we wish to solve the same IK problem as in Eq. 12. If the axes of ξ_1 and ξ_2 intersect at a point q, we can select a point p that is not on the axis of ξ_3 and simplify to the following:

$$\begin{aligned}\delta := ||gp - q|| &= ||e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}e^{\widehat{\xi_3}\theta_3}p - q|| \\ &= ||e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}(e^{\widehat{\xi_3}\theta_3}p - q)|| \\ &= ||e^{\widehat{\xi_3}\theta_3}p - q||\end{aligned} \tag{13}$$

which is just Subproblem 3. In this way, we have eliminated the exponentials $e^{\widehat{\xi_1}\theta_1}, e^{\widehat{\xi_2}\theta_2}$ from the *left* hand side of the product of exponentials. Once we solve for θ_3, the matrix $e^{\hat{\xi}_3\theta_3}$ will be known and we can return to finding the rest of the θ_i's.

3.1.3 Trick 3: Dealing with prismatic joints

The 3 PK subproblems are all only relevant when dealing with revolute joints, as they all describe scenarios where points are being rotated about axes. So in general, we will not use any predefined subproblems when dealing with prismatic joints. Rather, we will exploit the structure of the specific manipulator.

Note: It is usually a good idea to try solving the prismatic joints first.

Often, prismatic joints directly control the distance between some two points p and q. This happens when the vector $p - q$ is parallel to the axis of the prismatic joint. When this is the case, we should try to reduce the IK problem to the form

$$\left\| e^{\hat{\xi}\theta}p - q \right\| = \delta$$

where ξ is the unit twist corresponding to the prismatic joint. This can usually be done with a combination of trick 1 (for picking p) and trick 2 (for picking q). In this case, θ is exactly the distance by which p gets moved, which allows us to find θ as simply the difference between δ and the starting distance $||p - q||$, taking care of the sign when necessary.

In other instances, such as the SCARA example from later in this discussion worksheet, the prismatic joint directly controls one of the coordinates of the end effector. There again, it is easy to compute θ by inspection by considering that coordinate of the desired configuration.

4 Elbow manipulator example

Break down the inverse kinematics for the elbow manipulator in Fig. 4 into simpler PK subproblems.

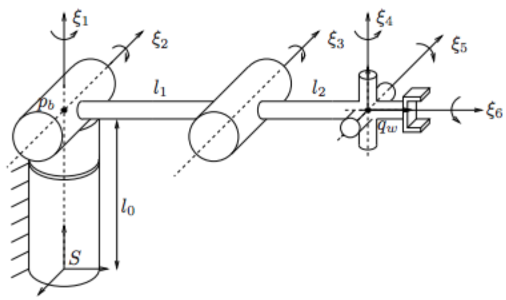

Figure 4: Elbow manipulator.

The inverse kinematics problem can be written as

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6} = g_d g_{st}^{-1}(0) := g$$

where we have defined g to be the known matrix $g_d g_{st}^{-1}(0)$.

Step 1: Eliminate $\theta_4, \theta_5, \theta_6$ using trick 1 and θ_1, θ_2 using trick 2. Take q_w as a point common to joints 4,5,6 and p_b as a point common to joints 1 and 2. Multiplying by q_w gives us

$$\begin{aligned}
&e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6} = g \\
\implies &e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}q_w = gq_w \\
\implies &e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_w = gq_w
\end{aligned}$$

we still have 3 joints, so to use our subproblems we need to eliminate some more variables. We now subtract off q_b to eliminate θ_1 and θ_2

$$\begin{aligned}
&e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_w - q_b = gq_w - q_b \\
\implies &\left\| e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\left(e^{\hat{\xi}_3\theta_3}q_w - q_b\right)\right\| = \|gq_w - q_b\| \\
\implies &\left\| e^{\hat{\xi}_3\theta_3}q_w - q_b\right\| = \|gq_w - q_b\|
\end{aligned}$$

this is exactly the set up for subproblem 3. So we use subproblem 3 to solve for θ_3. Once we do so, θ_3 is known and so is the matrix $g_3 = e^{\hat{\xi}_3\theta_3}$. The IK problem is now written as

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2} \cdot g_3 \cdot e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6} = g$$

Step 2: Solve for θ_1, θ_2 by eliminating joints 4,5,6. Once again, we pick q_w and multiply both sides of the above to get

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2} \cdot g_3 q_w = gq_w$$

Since g_3 is known, this is exactly the set up of subproblem 2 with the intersecting axes of joints 1 and 2 and the known points $g_3 q_w$ and $g q_w$. Use suproblem 2 to solve for θ_1, θ_2. Now that both $g_1 = e^{\hat{\xi}_1 \theta_1}, g_2 = e^{\hat{\xi}_2 \theta_2}$ are also known. So we can move g_1, g_2, g_3 to the right hand side to get

$$e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} e^{\hat{\xi}_6 \theta_6} = g_3^{-1} g_2^{-1} g_1^{-1} g := g'$$

where we have defined g' to be the known matrix $g_3^{-1} g_2^{-1} g_1^{-1} g$ for convenience.

Step 3: Solve for θ_4, θ_5 by eliminating θ_6. Now, we need to pick some point q that is on the axis of joint 6, but NOT on the axis of either joints 4 or 5 (otherwise they would get eliminated too and we would not be able to solve for them). We get

$$e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} q = g' q$$

which is exactly subproblem 2. So we use that subproblem to solve for θ_4 and θ_5. Now the matrices $g_4 = e^{\hat{\xi}_4 \theta_4}, g_5 = e^{\hat{\xi}_5 \theta_5}$ are also known, so we can move them to the right hand side

$$e^{\hat{\xi}_6 \theta_6} = g_5^{-1} g_4^{-1} g' := g''$$

where we have defined g'' to be the known matrix $g_5^{-1} g_4^{-1} g'$.

Step 4: Finally, we can use subproblem 1 to solve for θ_6. Pick any point p not on the axis of joint 6. Then we have

$$e^{\hat{\xi}_6 \theta_6} p = g'' p$$

which is subproblem 1. We have now solved for all the θ_i's, and we are done.

5 SCARA manipulator example

Break down the the inverse kinematics for the SCARA manipulator in Fig. 5 into simpler PK subproblems.

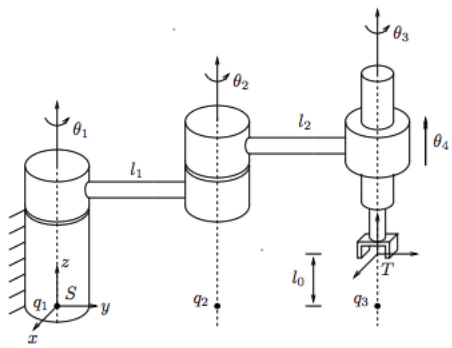

Figure 5: SCARA manipulator.

The inverse kinematics problem can be written as

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4} = g_d g_{st}^{-1}(0) := g$$

where we have defined g to be the known matrix $g_d g_{st}^{-1}(0)$.

Step 1: We should start off by solving for the prismatic joint, since we do not have subproblems to handle those. In this case, we can notice that θ_4 directly controls the z coordinate of the origin of frame T. So if the z coordinate of g_d is some p_z, then we require that $l_0 + \theta_4 = p_z$, and so $\theta_4 = p_z - l_0$. Now θ_4 is known, and so is the matrix $g_4 = e^{\hat{\xi}_4\theta_4}$. So we bring this matrix over the right hand side, and re-write the IK problem as

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3} = g g_4^{-1} := g'$$

where we have defined g' to be the known matrix $ge^{-\hat{\xi}_4\theta_4}$ for convenience.

Step 2: Next, let's try to eliminate θ_3. We can pick point q_3, as that is on the axis of joint 3. We get

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_3 = g'q_3$$
$$\iff e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}q_3 = g'q_3$$

This looks a lot like subproblem 2, but careful! It is not a valid instance of subproblem 2, since the axes of joints 1 and 2 do not intersect! So we are not yet done. Instead, we should also eliminate θ_1 by subtracting off a point that is on the axis of joint 1 and then taking the norm. We pick q_1 as that point. Then we can write

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}q_3 - q_1 = g'q_3 - q_1$$
$$\implies ||e^{\hat{\xi}_2\theta_2}q_3 - q_1|| = ||g'q_3 - q_1||$$

which is a valid instance of subproblem 3. So we can use subproblem 3 to find θ_2. Now θ_2 is known, and so is the matrix $g_2 = e^{\hat{\xi}_2\theta_2}$. We cannot move this matrix to the RHS just yet because it is sandwiched between two unkown exponentials, we can keep track of the fact that it is known. The IK problem is now written as

$$e^{\hat{\xi}_1\theta_1} \cdot g_2 \cdot e^{\hat{\xi}_3\theta_3} = g'$$

Now we will just solve for θ_1 and θ_3 in turn.

Step 3: Solve for θ_1 by eliminating θ_3. Once again, pick q_3 as it is on the axis of joint 3, and multiply the above equation with it

$$\begin{aligned} e^{\hat{\xi}_1\theta_1} \cdot g_2 \cdot e^{\hat{\xi}_3\theta_3} q_3 &= g' q_3 \\ \implies e^{\hat{\xi}_1\theta_1} \cdot (g_2 q_3) &= g' q_3 \end{aligned}$$

Since g_2 is known, this is an instance of subproblem 1 with the known point $g_2 q_3$ being rotated onto $g' q_3$. So we can use subproblem 1 to solve for θ_1. Now, the matrix $g_1 = e^{\hat{\xi}_1\theta_1}$ is also known, and we can move both g_1 and g_2 to the right hand side

$$e^{\hat{\xi}_3\theta_3} = g_2^{-1} g_1^{-1} g' := g''$$

where we have defined g'' to be the known matrix $g_2^{-1} g_1^{-1} g'$.

Step 4: Use subproblem 1 to solve for θ_3. Pick any point p not on the axis of joint 3. Then multiply the above with that point to get

$$e^{\hat{\xi}_3\theta_3} p = g'' p$$

this is a valid instance of subproblem 1, so we can use it to solve for θ_3. Now all θ_i's are known and we are done.