EECS106A Discussion 1: Rotations

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1 Frame-specific representations

Points and vectors are described by coordinates that are only meaningful with respect to a corresponding coordinate frame.

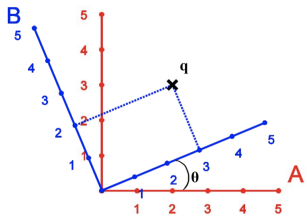

Figure 1: Two coordinate frames A and B

Problem 1. *Write the representation of point q with respect to the coordinate frames A and B, which we denote q_a and q_b respectively.*

$$q_a = [2 \quad 3]^T, q_b = [3 \quad 2]^T$$

2 Rotation matrices

Let's first think solely about the mathematical definition of a rotation matrix before discussing how they are used in practice. A rotation matrix is a matrix that is defined according to two coordinate frames.

Definition 1. Say we have coordinate frame A, defined by its principal axes $\{\boldsymbol{x_a}, \boldsymbol{y_a}, \boldsymbol{z_a}\}$, and frame B, with principal axes $\{\boldsymbol{x_b}, \boldsymbol{y_b}, \boldsymbol{z_b}\}$. Then, we define a rotation matrix R_{ab} to be

$$R_{ab} := [\boldsymbol{x_{ab}} \quad \boldsymbol{y_{ab}} \quad \boldsymbol{z_{ab}}]$$

where $\{\boldsymbol{x_{ab}}, \boldsymbol{y_{ab}}, \boldsymbol{z_{ab}}\}$ are orthonormal principal axes of frame B expressed in the coordinates of frame A.

Problem 2. *Find the rotation matrix $R_{ab} = [\boldsymbol{x_{ab}} \quad \boldsymbol{y_{ab}}]$ for an arbitrary 2D rotation (as depicted in Fig. 1)*

$$R_{ab} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

In general in 3D, we have three elemental rotation matrices that arise from rotations either about the x, y, or z-axis.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}; \quad R_y(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

Problem 3. *Work out what $R_z(\theta)$ is.*

$$R_z = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 Uses of rotation matrices

3.1 Representing the orientation of a frame

This follows from the definition of a rotation matrix above. For any pair of coordinate frames A and B, there exists *one* unique rotation matrix R_{ab}— thus, R_{ab} tells us exactly how frame B is oriented from the reference of frame A.

Problem 4. *Find the rotation matrix R_{ba} given the same 2D coordinate frames in Fig. 1. What do you notice about the relationship between R_{ab} and R_{ba}?*

$$R_{ba} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

We notice that R_{ab} and R_{ba} are the transpose of each other.

Commutative rule: Say we now have three frames A, B, and C. I tell you what R_{bc} is, (ie. the orientation of C from the reference of B), and you've already calculated R_{AB}. How can we express R_{ac}, that is, the orientation of C from the reference of A? We simply combine rotation matrices to form a new rotation matrix through matrix multiplication:

$$R_{ac} = R_{ab}R_{bc}$$

3.2 Changing the reference frame

Rotation matrices can also be used to change the reference frame of a point or vector from one frame to another using the following rule with q being the coordinates of this point or vector:

$$q_a = R_{ab}q_b$$

Problem 5. *Find the exact value of θ in Fig. 1.*

Plugging in the values for q_a and q_b and solving the system of equations, we get that $\theta \simeq 0.345$ rad.

3.3 Transforming a point or vector in a fixed frame

Say you have a point or vector q in frame A that you want to rotate by some θ about an axis $\boldsymbol{\omega}$. If the entire of coordinate frame A was rotated by exactly this θ about $\boldsymbol{\omega}$, call the resulting orientation frame B. This induces a rotation matrix R_{ab}, and the transformed q, which we denote by q', is expressed by:

$$q'_a = R_{ab}q_a$$

Problem 6. *Given a point $q = (x, y)$, what are its new coordinates $q' = (x', y')$ after a rotation by a general θ counter-clockwise about the origin?*

$$x' = xcos\theta - ysin\theta$$

$$y' = xsin\theta + ycos\theta$$

4 Properties of rotation matrices

1. Columns of R are mutually orthonormal, ie. $RR^T = R^TR = I$
2. $\det R = \pm 1$ (+1 for right-handed frames)

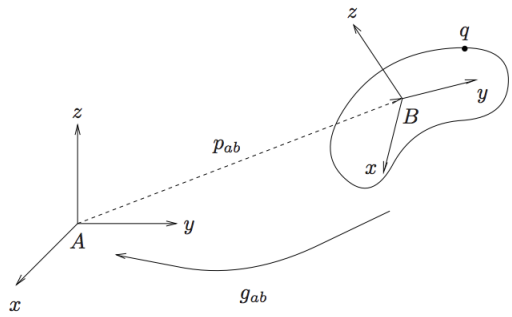

Figure 2: Illustration of a general rigid transformation between two frames.

5 Rigid body transformations

Definition 2. A mapping $g : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be a *rigid body transformation* if it satisfies the following properties:

1. Length is preserved: $\forall$ points $p, q \in \mathbb{R}^3$, $||p - q|| = ||g(p) - g(q)||$

2. The cross product (and therefore orientation) is preserved: $\forall$ vectors $v, w \in \mathbb{R}^3$, $g(v \times w) = g(v) \times g(w)$

Problem 7. *Complete the mathematical statements for these two properties of rigid body transformations above.*

Proposition. A rotation matrix when applied as an operator via matrix multiplication is a valid rigid body transformation.

In general, rigid body transformations consist of rotation and translation as is depicted in Fig. 2.

5.1 Rigid transformation of a point

$$q_a = p_{ab} + R_{ab} q_b$$

So the rigid transformation g on a point q is

$$g(q) = p + R(q)$$

This is an example of an *affine* transformation.

Definition 3. An affine transformation is a mapping $f : X- > Y$ of the form $x- > Mx + b$ where M is a linear transformation on the space X and b is a vector in the space Y.

5.2 Rigid transformation of a vector

Given a vector $v = s - r$, we have the rigid transformation

$$g(v) = g(s - r) = g(s) - g(r) = R(s - r) = R(v)$$

so the rigid transformation of a vector consists of just a rotation.

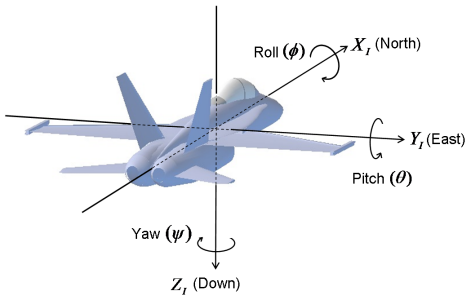

Figure 3: Roll, pitch, and yaw angles can describe orientation through rotations about the fixed coordinate axes.

6 Homogeneous coordinates

It's not the most convenient to represent a rigid transformation by both a rotation matrix and a translation vector, so we can introduce homogeneous coordinates that helps simplify the representation. We now represent points with an extra 1 appended, and vectors with a 0 appended. For example, points and vectors in $\mathbb{R}^3$ are transformed to be in $\mathbb{R}^4$:

$$\bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix}; \quad \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

Now, a rigid transformation can be expressed in a purely *linear* transformation.

$$\bar{q}_a = \begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix} =: \bar{g}_{ab}\bar{q}_b$$

$$\bar{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

7 Other representations of rotations

7.1 Axis angle

Any rotation can be expressed as a rotation of θ about a unit axis ω.

7.2 RPY angles

Rotations are described by three angles (roll ϕ, pitch θ, yaw ψ) about the basis vectors of a fixed coordinate frames. It involves the following intermediate rotations of a body frame B about a fixed world frame A, where B and A are initially coincident:

- Rotate A about its x-axis of A by the roll angle ϕ. Call this new frame B.
- Rotate B about the y-axis of A by the pitch angle θ. Call this updated frame C.
- Rotate C about the z-axis of A by the yaw angle ψ. Call this new and final frame D.

Thus, by the composition of these transformations, the final resultant rotation matrix is

$$R_{ad} = R_z(\psi)R_y(\theta)R_x(\phi)$$

This is an example of *extrinsic* rotations— ones that are all defined with respect to a fixed frame.

7.3 Euler angles

Euler angles describe the rotations about the changing basis vectors. For example, these are the rotations involved in what we call the ZYX Euler angles, here denoted as (α, β, γ). Let frame A be the initial orientation of the object being rotated.

- Rotate A by α about the z-axis. Call this new frame B.
- Rotate B by its *new* y-axis by β. Call this new frame C
- Rotate C by its *new* x-axis by γ. Call this new and final frame D.

The final resultant rotation matrix is derived from the composition of these three intermediate rotations.

$$R_{ad} = R_{ab}R_{bc}R_{cd} = R_z(\alpha)R_y(\beta)R_x(\gamma)$$

This is an example of *intrinsic* rotations— ones that are all defined with respect to the rotating coordinate frame.

7.4 Relationship between intrinsic and extrinsic rotations

It turns out that extrinsic rotation is equivalent to an intrinsic rotation by the same angles but with inverted order of elemental rotations, and vice-versa. Thus, a RPY transformation with roll, pitch, and yaw angles of (γ, β, α) is equivalent to the ZYX Euler angle rotations of (α, β, γ).

7.5 Quaternions

"Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way..." — W. Thompson, Lord Kelvin. (1892).
The mathematics of quaternions is outside the scope of this course. However, unit quaternions are very useful for encoding 3D rotations, and you'll be seeing them a lot in lab. A unit quaternion Q is a vector with four components: x, y, z, and w, such that $\|Q\| = 1$. (Different software may represent quaternions as WXYZ or XYZW; watch out for that!) You can find these terms from an axis angle representation as follows:

$$w = \cos\left(\frac{\theta}{2}\right) \quad x = \omega_1 \sin\left(\frac{\theta}{2}\right) \quad y = \omega_2 \sin\left(\frac{\theta}{2}\right) \quad z = \omega_3 \sin\left(\frac{\theta}{2}\right)$$

where ω is a unit vector along the axis of rotation, and θ is the angle of rotation.

Benefits over Euler angles

- Represent SO(3) without singularities

Benefit over Rotation Matrices

- Only requires four values, rather than 9.
- Quaternion multiplication is much faster than matrix multiplication.

Problem 8. *Prove that any quaternion generated from a unit ω and an arbitrary θ will be a unit quaternion.*

$$\begin{aligned}
||Q|| &= \omega^2 + x^2 + y^2 + z^2 \\
&= cos^2\left(\frac{\theta}{2}\right) + \omega_1^2 sin^2\left(\frac{\theta}{2}\right) + \omega_2^2 sin^2\left(\frac{\theta}{2}\right) + \omega_3^2 sin^2\left(\frac{\theta}{2}\right) \\
&= cos^2\left(\frac{\theta}{2}\right) + sin^2\left(\frac{\theta}{2}\right) ||\omega||^2 \\
&= cos^2\left(\frac{\theta}{2}\right) + sin^2\left(\frac{\theta}{2}\right) \\
&= 1
\end{aligned}$$

8 Rodrigues' formula

What if we don't want to use the elemental rotation matrices R_x, R_y, R_z? To express a general rotation about some axis ω with $||\omega|| = 1$ by some angle θ, we utilize Rodrigues' formula to extract the resulting rotation matrix:

$$R = I + \hat{\omega} sin\theta + \hat{\omega}^2(1 - cos\theta)$$

where the ˆ operator transforms a vector into its skew symmetric matrix as such:

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}; \quad \hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix};$$